Math Research Experience: Surface Intersections

Zane Li

1 Research Experience

During the summer between my junior and senior year, I was extremely lucky to be one of the 80 rising seniors who were admitted to the Research Science Institute (RSI) held at MIT. My mentor was Ryan Reich, a graduate student from Harvard University. Before the program, I was contacted by my mentor and was given a couple of project ideas. I could either do a project on tropical geometry or do a project similar to that of his previous student. I chose the latter. His previous student studied the intersection of two quadric surfaces[1] in $\mathbb{CP}^3$ and found an explicit normal form[2]. This paper can be found at [9].

I thoroughly read through the paper and learned of things such as projective space, pencils, fractional linear transformations, and birational equivalence. After I finished reading this paper, I decided to read some of its references. I read a few chapters from Silverman and Tate's *Rational Points on Elliptic Curves* ([6]) and Silverman's *Arithmetic of Elliptic Curves* ([7]), and a couple of papers that were used by Q. Yuan in his paper such as [1], [4], [5]. Additionally I decided to learn some complex analysis from Brown and Churchill's *Complex Analysis and Applications* ([2]). Through this I familiarized myself with elliptic curves and their group law, quadric surface intersection, the Weierstrass form of elliptic curves, and some more projective geometry. This whole process of learning more math took about a month before I left for RSI.

[1]A quadric surface and a cubic surface are defined at the beginning of Section 3.

[2]A normal form X for a certain set of objects S means that every element in S is equivalent to X. For example, $(x-h)^2+(y-k)^2=r^2$ is a normal form for all circles.

While at RSI, I initially tried to find an explicit normal form for the intersection of a quadric surface and a cubic surface by trying to adapt Q. Yuan's results to mine. However, after a few weeks of no success, I decided to adapt the methods of [8] instead to the intersection of quadric and cubic surfaces. Since I had only time to do four weeks of research at RSI, after some initial success of adapting Wang et al.'s methods to the intersection of a quadric surface with a cubic surface, RSI ended. However, since math is rather portable, I continued to think about my problem when I got home.

Since I was trying to adapt the methods of [8], I would be trying to find a parameterization of the intersection curve. I hit numerous impasses and sometimes spent hours at my local university's library thinking and looking for theorems and tools. For example, when I started to parameterize my space curve, I ran into the problem that one of Wang et al.'s theorem that was crucial in parameterization failed for my intersection space curve projection. I was stuck, but I knew I could adapt it some way. I read ahead. I tried multiple ways of attacking this problem. Many times I failed. I didn't give up. I just went back to my notes, and looked for new ideas that I had written down. Weeks later, while I was reading the parameterization section in Wang et al.'s paper, I suddenly realized that if I lowered the degree of the space curve projection, I could successfully adapt the failed theorem. Another example would be that initially I didn't have the Taylor polynomial for a multivariate function that was needed for parameterizing the space curve. I pored over all the calculus and analysis books that I had and my local library had. Finally, I was able to track this definition to Colley's *Vector Calculus* ([3]). Finding this formula took days, I could finally proceed on with my research.

Eventually I was able to find the parameterization for a special case of the intersection curve. This is what I wrote my paper on. I eventually used this paper to enter the Siemens Competition and the Intel Science Talent Search and achieved Semifinalist status in both competitions. This research experience gave me a glimpse of math in a non-school environment and made me want to become a mathematician even more than before.

In the following section, I will give a brief introduction to some projective geometry. My research summary is in Section 3.

2 Some Basic Projective Geometry

In this section I summarize some of the tools I used in my research. This section contains some basic terms and definitions along with a statement of Bézout's Theorem. There is a more thorough introduction to projective geometry in Appendix A of [6].

In my paper we work in $\mathbb{CP}^3$, that is, complex projective 3-space. We first begin by defining homogeneous coordinates. Points in $\mathbb{CP}^3$ are homogeneous coordinates written like $[\ a_1 \quad a_2 \quad a_3 \quad a_4\ ]$ where $a_i \in \mathbb{C}$. We say that two points $[\ a_1 \quad a_2 \quad a_3 \quad a_4\ ]$ and $[\ b_1 \quad b_2 \quad b_3 \quad b_4\ ]$ are equivalent if there is a non-zero λ such that $a_i = \lambda b_i$ for each i. For example the points $[\ 1 \quad 2 \quad 3 \quad 4\ ]$ and $[\ 3 \quad 6 \quad 9 \quad 12\ ]$ are equivalent in projective space.

We now define $\mathbb{CP}^3$, that is complex projective 3-space. Complex projective 3-space is defined to be the set of all quadruples $[\ a_1 \quad a_2 \quad a_3 \quad a_4\ ]$ with $a_1, a_2, a_3, a_4 \in \mathbb{C}$ and not all equal to 0 and where each quadruple is not equivalent to any other quadruple in the set. We now define $\mathbb{A}^3$, that is affine 3-space. Affine 3-space is defined to be the set of triples (a_1, a_2, a_3) with a_i any number.

A homogeneous polynomial is a polynomial all of whose terms have the same degree. For example, $x^3y - x^2y^2 = 0$ is a homogeneous polynomial while $x^2 - y = 0$ is not. We can dehomogenize a homogeneous polynomial by setting the variable we want to dehomogenize by to be 1. For example, if we dehomogenize the homogeneous polynomial $XW^2 - Z^2W = 0$ by W, we get the polynomial $x - z^2 = 0$. We can also dehomogenize a point in projective geometry, in other words we can switch from projective points to affine points. Assume we have the projective coordinate $[\ a_1 \quad a_2 \quad a_3 \quad a_4\ ]$ in $\mathbb{CP}^3$. We dehomogenize by a_3 and we get $\begin{bmatrix} \frac{a_1}{a_3} & \frac{a_2}{a_3} & 1 & \frac{a_4}{a_3} \end{bmatrix}$. Now we have the coordinate $\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}, \frac{a_4}{a_3}\right)$ in the affine space.

The plane at infinity in $\mathbb{CP}^3$ is given by the projective coordinates $[\ r \quad s \quad t \quad 0\ ]$. We notice that if we dehomogenize by the last coordinate, we are dividing by 0. A property of the plane at infinity is that parallel lines intersect at the plane at infinity.

Additionally, the following theorem is used in my research.

Theorem (Bézout's Theorem). *In projective space, if there are two surfaces, one of degree m, and another of degree n, then the intersection curve has degree mn. Similarly, if there is a surface of degree m and a curve of degree n, then the curve and the surface intersect at mn points.*

3 Research Summary

The general homogeneous polynomial for a quadric surface in $\mathbb{CP}^3$ can be written as

$$Q(X,Y,Z,W) = \mathbf{X}Q_1\mathbf{X}^T = 0,$$

where Q_1 is a complex, invertible, symmetric 4×4 matrix and where $\mathbf{X} = \begin{bmatrix} X & Y & Z & W \end{bmatrix}$ is in homogeneous coordinates. Similarly, the general homogeneous polynomial for a cubic surface in $\mathbb{CP}^3$ can be written as

$$C(X,Y,Z,W) = X\mathbf{X}C_1\mathbf{X}^T + Y\mathbf{X}C_2\mathbf{X}^T + Z\mathbf{X}C_3\mathbf{X}^T + W\mathbf{X}C_4\mathbf{X}^T = 0,$$

where

$$C_1 = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}, \quad C_2 = \begin{bmatrix} b & e & f & g \\ e & k & l & m \\ f & l & n & q \\ g & m & q & p \end{bmatrix}, \quad C_3 = \begin{bmatrix} c & f & h & i \\ f & l & n & q \\ h & n & q & r \\ i & q & r & s \end{bmatrix}, \quad C_4 = \begin{bmatrix} d & g & i & j \\ g & m & q & p \\ i & q & r & s \\ j & p & s & t \end{bmatrix}.$$

We first parameterize the quadric surface in a similar manner demonstrated in [8], that is, let $\mathbf{X_0}$ be a fixed point on the quadric surface Q, and let $\mathbf{T} = \begin{bmatrix} r & s & t & 0 \end{bmatrix}$ be a point on the plane at infinity. Then

$$P(r,s,t) = (\mathbf{T}Q_1\mathbf{T}^T)\mathbf{X_0} - 2(\mathbf{X_0}Q_1\mathbf{T}^T)\mathbf{T}$$

is a parameterization of the quadric surface Q.

Define a space curve to be the intersection curve in space between a quadric surface and a cubic surface. Define a plane curve to be the space curve projected via $P(r,s,t)$ onto the (r,s,t) plane, the plane at infinity, through a point on the space curve. The space curve corresponds to a plane curve $\widehat{I}(r,s,t) = 0$, which is a certain polynomial of degree 6 by Bézout's Theorem. This polynomial $\widehat{I}(r,s,t) = 0$ can be further factored by using the following theorem.

Theorem. *We have* $\widehat{I}(r,s,t) = (\mathbf{X_0}Q_1\mathbf{T}^T)^k I(r,s,t)$, *where* $I(r,s,t)$ *is a degree* $6-k$ *polynomial if and only if* $\mathbf{X_0}$ *is a* k*-fold point*[3] *on the space curve between the quadric and cubic surface.*

Because we projected through the point $\mathbf{X_0}$ to obtain our parameterization, by a geometric argument, we can show that $\mathbf{X_0}Q_1\mathbf{T}^T = 0$. Hence the polynomial for the plane curve is

$$I(r,s,t) = \frac{\widehat{I}(r,s,t)}{(\mathbf{X_0}Q_1\mathbf{T}^T)^k}$$

if $\mathbf{X_0}$ is a k-fold point of the space curve.

Define a base point to be a point (α,β,γ) that is not $(0,0,0)$ which satisfies $P(\alpha,\beta,\gamma) = 0$. This point corresponds to the point $[\ \alpha \quad \beta \quad \gamma \quad 0\ ]$ on the plane at infinity in $\mathbb{CP}^3$.

Let $\mathbf{X_0}$ be a 3-fold point. By the previous equation, we see that $I(r,s,t)$ is a degree 3 equation. We can use methods similar in that of [8] to parameterize the intersection curve. Despite the limited scope of our parameterization theorems, as our parameterization theorems are rational or just contain a square root we provide a simple and efficient method

[3]A k-fold point on the space curve between the quadric and cubic surface means that if we intersect a plane with the curve through the k-fold point, the plane will have $6-k$ other intersections with the curve.

for finding the parameterization of the intersection curve in space.

Our main results are obtained by using the multivariate Taylor polynomial[4] for the polynomial $I(r,s,t)$. It is as follows

Definition. *(cf. [3]) The k-th order Taylor polynomial for function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ of class C^k near $\mathbf{a} \in X$ is given by*

$$p_k(\mathbf{x}) = f(\mathbf{a}) + \sum_{i_1=1}^{n} f_{x_{i_1}}(\mathbf{a})(x_{i_1} - a_{i_1}) + \frac{1}{2!}\sum_{i_1,i_2=1}^{n} f_{x_{i_1}x_{i_2}}(\mathbf{a})(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2}) \qquad (1)$$
$$+ \cdots + \frac{1}{k!}\sum_{i_1,i_2,\cdots,i_k=1}^{n} f_{x_{i_1}x_{i_2}\cdots x_{i_k}}(\mathbf{a})(x_{i_1} - a_{i_1})(x_{i_2} - a_{i_2})\cdots(x_{i_k} - a_{i_k}).$$

There are now two cases to consider for $I(r,s,t)$, when it is a singular curve and when it is a non-singular curve[5]. Note that the multivariate Taylor polynomial is for nonhomogeneous polynomials, so we must first dehomogenize $I(r,s,t)$ and then rehomogenize. By using the above definition, we obtain the following two results.

Theorem. *The space curve whose projection is an irreducible singular curve $I(r,s,t)$ has a degree 4 rational parameterization if $I(r,s,t)$ passes through both base points of $P(r,s,t)$. Moreover, this parameterization of the space curve is of the form*

$$R(u,v) = \frac{pNp^T}{(ub_0 - va_0)(ub_1 - va_1)}.$$

Theorem. *The space curve whose projection is an irreducible nonsingular curve $I(r,s,t)$ has a parameterization involving a square root if $I(r,s,t)$ passes through both base points of $P(r,s,t)$. Moreover, this parameterization is of the form*

$$R(u,v) = \frac{F(u,v)}{uv_0 - vu_0} \pm \frac{G(u,v)}{uv_0 - vu_0}\sqrt{D(u,v)}.$$

[4]Recall from single variable calculus that the Taylor polynomial for a function $f(x)$ near the point a is $\sum_{k=0}^{n} \frac{f^{(n)}(a)}{n!}(x-a)^n$. In short, the multivariate Taylor polynomial is a generalization of this.

[5]A singular curve has at least one singular point. A point (α,β,γ) is said to be singular if $I(\alpha,\beta,\gamma) = 0$ and all partial derivatives vanish at (α,β,γ). In other words $I(\alpha,\beta,\gamma) = 0$ and $\frac{\partial I}{\partial r} = \frac{\partial I}{\partial s} = \frac{\partial I}{\partial t} = 0$ when evaluated at (α,β,γ)

These two theorems partially extend the results of [8] to the intersection of quadric and cubic surfaces. One possible continuation of this research could be to parameterize the intersection curve when $I(r, s, t)$ is a quintic polynomial. This would imply that $\mathbf{X_0}$ is a 1-fold point, which is the most general case. By doing this, we would have completely adapted Wang et. al's methods to the intersection of quadric and cubic surfaces. Another possible continuation would be to find an explicit normal form for the intersection of a quadric and a cubic surface, thus extending the results of [9].

References

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