My STS project “Deligne categories and representation theory in complex rank”

Akhil Mathew

July 5, 2010

1 My involvement in research

The summer after my junior year, I went to the Research Science Institute (RSI) program at MIT. I had a blast there, and I strongly encourage any eligible students reading this to apply. I had two mentors: a graduate student named Dustin Clausen and a professor named Pavel Etingof. My mentors contacted me before the program to tell me about a potential project on representation theory in complex rank, following a paper [1] of Deligne that laid the groundwork and beginning work on a program that Etingof himself had proposed in a talk at the Newton Institute.

There were a few obstacles. First, Deligne writes in French. It’s a good thing that I take the language in school, but I’m not terribly fluent. Fortunately, mathematicians tend not to use difficult words; most of the technical math jargon consists of cognates anyway. Recognizing “catégorie” as “category” does not require translator-level skills.

A more serious difficulty was that Deligne’s paper is hard. Academic math papers in general have a tendency to focus on correctness over understandability (the word "trivial" is used very differently by research mathematicians and other people, for instance). Deligne's paper also heavily uses the language of *category theory*, a branch of mathematics whose dryness has earned it the nickname "abstract nonsense" among mathematicians. I still don't understand at least half of *La catégorie des représentations du groupe* S_t *lorsque* t *n'est pas un entier naturel.*

Nevertheless, I did manage to assimilate a few facts about monoidal categories and get a sense of the basic constructions by the time the program started in late June 2009. In the meantime, I had also been reading a book [4] (in French, incidentally) by A. Grothendieck titled *Elements de Géometrie Algébrique* . The book is on algebraic geometry, and it was supposed to be a thirteen-volume treatise covering the main techniques, except only the first four volumes (which in total are over 1000 pages of reader-unfriendly mathematics in a small font) were published. I was reading volumes[1] 0 and 1 to try to get some understanding of this subject, and as a tool for procrastination—my project wasn't supposed to be about it.

Anyway, I met regularly with my mentors during RSI, yet the only bit of real progress I made then was to think about the fact that a one-variable polynomial has finitely many roots, unless it is identically zero. This elementary fact was what I used to try to prove a basic result about some of the objects Etingof had defined. It turned out that my reasoning was entirely wrong, although I didn't know it then.

After RSI, which is when I did almost all of the real work, I was rather confused.[2] My mentors said I should try to apply algebraic geometry and suggested an approach that I was looking into, but after a fair bit of looking during sweaty,

[1] It is a tradition among mathematicians to refer to prerequisites in sections (or chapters, or volumes) numbered 0.

[2] The reason my project has the rather loose title "Deligne categories and representation theory in complex rank" is that the Siemens deadline required me to submit a title before my project was ready. I didn't change it for Intel.

long walks on August afternoons, it seemed there was a small mistake that ruined everything. So I had to go in a different direction, after which it became clear to me that what I did during RSI was wrong. It was now August, and the Siemens competition deadline (September 31st) was looming in two months. The Siemens competition is near and dear to my mentor's heart—he told us a story during RSI about a student three years back who won the competition with an incredibly advanced project, judged by the nation's best topologist.

I spent that September with no life—I devoted almost every waking moment to figuring something out. But in the end, I did. It was probably the messiest and most unpleasant paper that I ever will put my name on, but there it was—squeezed within the maximum 18 pages by a combination of creative font selection,[3] liberal use of footnotes, and repetition of the William Strunk mantra "Omit needless words!" Basically, I resurrected an idea of an algebro-geometric parametrization of the objects of interest that had been discarded a while back, and put it to good use.

After the Siemens competition deadline passed, I went about reading two papers of Knop (cf. [6] and [5]; readers that are not masochists should probably start with [6]), that basically took the abstract nonsense in Deligne's to dizzying new heights. Nevertheless, the paper was written in an unusually limpid style that made you feel that the author really wanted to be understood. At least, it helped clarify what was going on in Deligne, and I enjoyed it. All the same, I didn't make any clear progress on actual *research*, and my paper for the Intel Science Talent Search (whose deadline is in November) wasn't terribly different from the Siemens one, except that I had to list my SAT scores and class rank and write various essays; it was rather like a college application.

I didn't work on my project continuously, and I devoted most of the next

[3]Tip for future science fair-letes: Times New Roman is better than Computer Modern (the LaTeX default).

few months to learning about a branch of mathematics known as differential geometry (this includes all sorts of generalizations of classical euclidean geometry studied in school; it is the mathematics used in general relativity, for instance), understanding some of its role in the theory of partial differential equations,[4] as well as finishing college applications. In short, I took off for a couple of months.

Then I got the phone call from Elizabeth Marincola in January telling me that I was a finalist, which put Deligne categories back on my mind. While preparing for the Intel Science Talent Institute,[5] I stumbled into a few extensions of my previous work. So the work I presented at STS was different (larger) from the paper I submitted there for selection as a finalist.

2 What I actually did (sort of)

I'm going to say right away that if you want full technical details, you should read my paper [7], which I have also discussed and explained at my blog (cf. [8], for instance). The story of my project starts with an algebraic object that shows up repeatedly, called the symmetric group and denoted S_n, where n here stands for a positive integer. S_n denotes the collection of ways you can rearrange a set containing n elements. More formally, an element of S_n is a function $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ which is a one-to-one correspondence. The important algebraic structure on S_n comes from the fact that you can *compose* such elements, because you can compose functions. This *law of composition* makes S_n into what mathematicians call a *group.* This is not the place for too much abstract formalism (or nonsense) here, but a group is basically a general notion of a set with a law of composition satisfying certain constraints. For instance, the real numbers $\mathbb{R}$ form a group if "composition" is taken as

[4] A partial differential equation is one that involves partial derivatives: for instance, the **Laplace equation** $\frac{\partial^2}{\partial^2 x}u + \frac{\partial^2}{\partial^2 y}u = 0$ of great importance in mathematical physics.

[5] This is where the forty finalists present their projects.

addition.

Groups, however, are very complicated in general. The classification of the finite *simple*[6] groups[7] runs to something like 10,000 pages. That's why mathematicians like to reduce group theory to linear algebra, that branch of mathematics that studies things like matrices

$$\begin{bmatrix} 4 & 2 & -5.5 \\ \pi & 3-2i & 5\sqrt{2} \end{bmatrix}.$$

Linear algebra is an introductory undergraduate course, so research mathematicians would call it trivial. Compared to 10,000 page proofs, it is.

So, how does this transformation of group theory into linear algebra work? Well, the point is that just as groups have a law of composition on them, matrices have something like composition: matrix multiplication.[8] The representation theory of finite groups is about correspondences between the vastly general, forbiddingly ivory-tower idea of laws of composition on abstract groups and the familiar, reassuring voice of our math teacher explaining how to multiply matrices.

For instance, what is a representation of S_n? It consists of an assignment of a m-by-m matrix M_σ to each σ in S_n, such that

$$M_\sigma \times M_\tau = M_{\sigma\circ\tau}$$

for all σ, τ. I've defined something, but we should all be skeptical about definitions without motivation. I can't give you an algebraic context for this here, but I can explain a cool fact. So, let's recall (or learn) that the trace of a

[6]"Simple" in mathematical parlance does not mean easy. Its actual meaning is closer to "like an atom" or "irreducible."

[7]This, incidentally, is the source of the hit love song "Finite simple group of order two."

[8]Matrices of a fixed square size do *not* form a group under multiplication, though, because not all matrices are invertible, and the group axioms require invertibility.

square matrix is the sum of the entries along the diagonal. Fix a representation $\sigma \to M_\sigma$ of S_n. Then it is a theorem that

$$n! \quad \text{divides} \quad \sum_{\sigma \in S_n} |\mathrm{Tr} M_\sigma|^2 .$$

The two quantities are equal if and only if the representation is *simple*, which means basically you can't split it up into two smaller representations. This is an elementary fact, which interested readers can learn from the course [3] (which my mentor gave to high school students[9] some years back).

Now I have to explain a revolution in mathematical thinking called *category theory* (a.k.a. abstract nonsense) that I mentioned earlier, and despite my biases, I will attempt to avoid evangelism. A *category* is basically a bundle of mathematical objects with relations between them. For instance, there is a category of groups. There's a category of representations of S_n. There's even a category of categories![10]

Why do we study categories? It turns out that many theorems can be proved for large classes of categories simultaneously, so the use of categories allows for a significant streamlining of mathematical thought. Then, category theory took on a life of its own. So we come to the paper of Deligne.

I mentioned that the representations of S_n could be packaged into one mathematical object, and this is a category; we call it $\mathrm{Rep}(S_n)$. It has quite a rich structure; the point is that the whole "representation theory of the symmetric group" mini-area of mathematics is basically the study of the category $\mathrm{Rep}(S_n)$, in some sense. Hold on, though. $\mathrm{Rep}(S_n)$ is really a *family* of categories, one for each $n = 0, 1, 2, \ldots$, and one for each symmetric group S_n.

[9]If you have difficulty with them, I sympathize—suffice it to say that it was a bit more than an honors course, though I didn't take it.

[10]Oh, and there are all sorts of set-theoretic traps lying around such meta-ideas; you have to hop like crazy to avoid being snapped up by Bertrand Russell.

What Deligne did was to define a category $\mathrm{Rep}(S_t)$ for t not necessarily an integer, but an arbitrary *complex number*. Granted, there's no such thing as a symmetric group S_π of permutations of a set with π elements, because nobody has ever seen a set with π elements. So this category is not the category of representations of anything. It's just a category to itself, floating in Platonic heaven.[11]

OK, you say. What's so special about Deligne's $\mathrm{Rep}(S_t)$? It's just a definition, so how do I know this category is not just some trivial object that you're dressing up in mathematical symbolism.[12] It turns out that, first of all, as Etingof explains in [2], the categories $\mathrm{Rep}(S_t)$ form an example of "tensor categories of superexponential growth." I won't even begin to explain any of that, but the point is that their existence shows that some of Deligne's earlier work says something that is not only true, but meaningful and interesting.

Moreover, while I can't explain how the categories $\mathrm{Rep}(S_t)$ are actually constructed here, the way they are is not purely random or ad hoc—it looks at the structure of the ordinary categories of representations of the symmetric groups S_n, identifies that many of the structures can be described via polynomials in the integer n, and instead substitutes an arbitrary t into those polynomials. Etingof explains the process in his talk; David Speyer does so in his blog post; and I do so in mine. It all boils down, again, to one-variable polynomials in the rank.

This idea of "interpolation" is not something that Deligne just pulled out of a hat. Earlier, Feigin had defined an algebra of "matrices of complex size"; this gives some meaning to the idea of "a π-by-π square matrix." These, however, are not categories; some other variants of Deligne's $\mathrm{Rep}(S_t)$ had been constructed by Deligne himself earlier because of their properties in the theory of tensor

[11] To use a phrase of Scott Aaronson.

[12] As the mathematical analog of Romeo once observed: What's in a symbol? $3SAT$ by another abbreviation would smell as intractable.

categories.

I've already mentioned Etingof's talk. It's actually about proposing a program of studying "representation theory in complex rank," when the symmetric group in Deligne's paper (and a few other variants) are replaced with a wide variety of algebraic structures that depend on integral parameters. He showed that it is possible to define their categories of representations in complex rank. In other words, what's been said above, but with the family $\{S_n\}$ replaced with something entirely different, for instance this family of objects called the *degenerate affine Hecke algebras*. His definitions build out of the categories defined by Deligne; basically, the point is that these Hecke algebras (among other things) are built out of the symmetric group, so their categories of representations (even when interpolated to complex rank) can be built similarly. Etingof leaves the viewer of his talk with the quest to study these categories.

My project was to begin working on this program. There are The Big Questions: namely, what Etingof calls "degeneracy phenomena." To use his colorful language, where do these new universes (representation theory in complex rank) that Etingof has discovered differ from our own, smaller, solar system (representation theory in integral rank) that we have already studied? Ultimately, the rather limited mathematical equipment I had aboard my spaceship (given my then being a high school student) was insufficient[13] for proper exploration into these dense corners. The project was, for the most part, well over my head, and I wasn't able to make serious progress on said phenomena. That is left for future, better-fueled voyagers.

I showed that these new universes are similar to our own solar system, and not even all that unfamiliar from Earth at home, *if* you make the rank a *transcendental* number.

[13]Perhaps Erika DeBenedictis, the winner of the 2010 STS competition, would disagree; cf. her project.

So, how did I do this? I parametrized these families of categories by certain *schemes*. I can't explain what a scheme is properly, but the point is that (as I show) the properties of these families of categories depend on the vanishing (or nonvanishing) of certain rational-coefficient polynomials at the rank t. I have no idea at all what these polynomials are, except that they exist.

And if such a polynomial with rational coefficients has no roots in the integers, it is nonzero and has no roots among transcendental numbers. Hence, representation theory in transcendental rank is similar to representation theory in integral rank (i.e., classical representation theory)—because a polynomial that vanishes on the nonzero integers can't vanish on the transcendentals.

Stripping away all the formalism and technical stuff, that's essentially what I did. I used more generality in the statements (that's how the scheme-theoretic business comes in); it creates a framework of "categories depending on a parameter" that may be usable in future investigation.

3 Acknowledgments

I thank Pavel Etingof and Dustin Clausen for mentoring me, and RSI for making it possible. I also thank Anirudha Balasubramanian for helpful comments on this paper.

References

[1] Pierre Deligne, *La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel*, Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces (2004).

[2] Pavel Etingof, *Representation theory in complex rank*, 2009, Conference talk at the Isaac Newton Institute for Mathematical Sciences.

[3] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Elena Udovina, and Dmitry Vaintrob, *Introduction to representation theory*, (2009), arXiv:0901.0827v3.

[4] Alexander Grothendieck and Jean Dieudonné, *Eléments de géometrie algébrique*, vol. 1, Springer-Verlag, 1971.

[5] Friedrich Knop, *A construction of semisimple tensor categories*, C. R. Math. Acad. Sci. Paris **343** (2006), 15–18, arXiv:math/0605126v2.

[6] ———, *Tensor envelopes of regular categories*, Advances in Mathematics **214** (2007), no. 2, 571–617, arXiv:math/0610552v2.

[7] Akhil Mathew, *Categories parametrized by schemes and representation theory in complex rank*, (2010), arXiv:1006.1381v1.

[8] ———, *Deligne's category* $\mathrm{Rep}(S_t)$ *for* t *not necessarily an integer*, 2010, Blog post, `https://amathew.wordpress.com/2010/06/08/delignes-category-reps_t-for-t-not-necessarily-an-integer/`.