Estimating Prime Power Sums

Francis Gerard
Mentored By: Lawrence Washington, University of Maryland

Personal Statement

Number theory is a field that has, for me, always held a special kind of magic. There is something about reading an elegant proof that sticks with me, gives me a certain feeling of gratification like the sensation experienced at the conclusion of a mystery novel. I have always believed that the objective of any mystery is to figure it all out before the grand reveal, to make the leaps of intuition before Sherlock Holmes. It is exactly that desire to investigate that compels me to study numbers. In number theory, when proving a theorem, you start with a problem, uncover clues, try out possibilities and *Eureka!* you've constructed a solution; case closed!

Now this eureka moment is not unique to number theory; certainly every field of study contains fulfilling epiphanies. However, number theory specifically appeals to me because you can take a very easily stated problem and find that what at first glance appeared to be a trivial inquiry, was in fact something inconceivably complex. Questions lead to more questions; it is a delicate balance of simplicity and impossibility, of order and chaos, constantly in flux. Even the most fundamental problems of number theory are not all that well understood because of this paradox. One such problem is trying to understand the distribution of prime numbers. Primes are very fundamental to the study of numbers because all natural numbers can be represented as a unique product of primes. Brilliant mathematicians, many of them personal heroes of mine such as Euler and Riemann, tried for years to make sense of these curious atoms of mathematics and in their years of study made monumental contributions. But the full truth has always remained elusive because primes are, in a word: chaotic. They remain a mathematical enigma to this day.

Hoping to make progress on some of these difficult problems, I reached out to Dr. Lawrence C. Washington, a University of Maryland professor with a passion for number theory and years of experience. He agreed to mentor me on whatever I wanted to study. The problem I chose to study was the $n^2 + 1$ prime problem that simply asks whether or not there are an infinite number of primes of the form $n^2 + 1$ (where n is a natural number). It was a very difficult problem that I didn't end up making much headway on, but I had to learn many new techniques such as sieve theory to even give the problem a serious attempt. During this time I learned one technique which interested me greatly: the technique of Riemann-Stieltjes integration. Dr. Washington showed me that it could be used with the prime counting function to rewrite certain sums. I was intrigued. So one day on the bus I started to play around with Stieltjes Integration. This playing around yielded some fascinating results which eventually became the subject of my paper.

Doing this project taught me that you don't have to be an Euler or a Riemann in order to make contributions to the field you love. In my experience, all you really need is passion and a bit of luck - possibly a superb mentor, as well. If I had to give some advice to young people looking to do research in any field that utilizes mathematics, it would be "love what you're doing." I believe that if you are passionate about something, you end up spending your time just doing that thing and immersing yourself in it. Sometimes you get lucky, and you get interesting results. Sometimes you don't, but you always learn something, and to me, that is a reward in itself.

Introduction

Prime Power Sums are sums of the form:

$$\sum_{p \leq x} p^n$$

where $n \in \Re$ and p is a prime number.

An example of this might be the sum of the primes less than 10. The prime power sum notation for this particular example would be:

$$\sum_{p \leq 10} p = 2 + 3 + 5 + 7 = 17$$

A more complex example might be the sum of the squares of the primes less than 10. The notation for that would be:

$$\sum_{p \leq 10} p^2 = 2^2 + 3^2 + 5^2 + 7^2 = 87$$

Such sums have been rigorously explored in the cases of $n = -1, 0, 1$. The case $n = -1$ is the harmonic prime series studied by Euler which he showed diverges as $ln(ln(x))$. The $n = 0$ case is the famous prime number theorem which aims to count the number of primes less than x (This can be easily seen because $p^0 = 1$ so it adds 1 for every prime). The $n = 1$ was explored by Bach and Shallit in 1996 in which they showed that the sum of primes less than x was asymptotic to $(x^2/2)ln(x)$.The interesting part about such sums is that they share a particular relationship with the prime counting function, a function which we will define to be $\pi(x)$ =the number of primes less than or equal to x. In our research, we generalize such sums by proving the following asymptotic formula:

Theorem: Let $n \geq 0$. Then

$$\sum_{p \leq x} p^n = \pi(x^{n+1}) + \mathcal{O}\left(\frac{\pi(x^{n+1})}{\ln(x)}\right)$$

This is important because as we will later see, it gives us a new method of estimating the prime counting function that allows you to use the primes up to one bound to estimate the number of primes up to a larger bound. This formula was devised while investigating the Riemann-Stieltjes integral which will be expounded upon at length in the next section.

Techniques

The technique that is most crucial in the proof of our main theorem is that of Riemann-Stieltjes Integration. Suppose we have some collection of points on the interval $a = x_0 < x_1 < x_2 < ... < x_n = b$ and the sum:

$$\sum_{i=0}^{n} f(x_i^*)(g(x_{i+1}) - g(x_i)))$$

Where x_i^* is some point on the interval $[x_{i+1}, x_i]$.

Then if the sum converges to a number N as $max(x_{i+1} - x_i) \to 0$, it can be approximated by the Riemann-Stieltjes Integral which is of the form:

$$\int_a^b f(x)d(g(x))$$

We are guaranteed the existence of the integral if $f(x)$ is continuous and $g(x)$ is of bounded variation. The utility of this statement in prime power sum estimation cannot be overstated.

This is because we can say the following:

$$\pi(x) - \pi(x-1) = \begin{cases} 1 & x \; is \; prime \\ 0 & x \; is \; composite \end{cases}$$

So if we are applying some function on primes less than x, then we can say:

$$\sum_{p \le x} f(p) = \sum_{n=2}^{x} f(n)(\pi(n) - \pi(n-1))$$

And by the Riemann-Stieltjes Integral we can say that:

$$\sum_{p \le x} f(p) = \sum_{n=2}^{x} f(n)(\pi(n) - \pi(n-1)) = \int_{2}^{x} f(x) d(\pi(x))$$

If $f(x)$ is continuous and differentiable (which implies $df(x) = f'(x)dx$) and the integrator, in this case $\pi(x)$, is of bounded variation (which it is), then we can integrate by parts. We can use this technique to evaluate the Riemann-Stieltjes Integral in our main theorem.

Proof of Main Theorem

Proof: To begin this proof we must consider the asymptotic expression for the function $\pi(x)$. As a consequence of the prime number theorem, we can express $\pi(x)$ as the following:

$$\pi(x) = \frac{x}{\ln(x)} + E(x), \quad E(x) = \mathcal{O}\left(\frac{x}{\ln^2(x)}\right)$$

The Stieltjes Integral then allows us to write our Prime Power Sum as:

$$\sum_{p \le x} p^n = \int_{1.5}^{x} t^n d(\pi(t))$$

Using the technique of integration by parts, we can rewrite our integral as:

$$= t^n\pi(t)\big|_{1.5}^{x} - n\int_{1.5}^{x} t^{n-1}\pi(t)dt$$

The evaluation of $t^n\pi(t)|_{1.5}^{x}$ simply is $x^n\pi(x)$ so we will substitute in the evaluation.

$$= x^n\pi(x) - n\int_{1.5}^{x} t^{n-1}\pi(t)dt$$

Now examining the second half of the expression for the sum, we realize that the integral contains a $\pi(x)$ term which we can naturally rewrite in its asymptotic form which was alluded to at the beginning of the proof. The result is below.

$$\int_{1.5}^{x} t^{n-1}\pi(t)dt = \int_{1.5}^{x} \frac{t^n}{\ln(t)}dt + \int_{1.5}^{x} t^{n-1}E(t)dt$$

Now taking the first half of that expression, we can integrate by parts again to obtain the following result.

$$\int_{1.5}^{x} \frac{t^n}{\ln(t)}dt = \frac{x^{n+1}}{(n+1)\ln(x)} + \frac{1}{n+1}\int_{1.5}^{x} \frac{t^n}{\ln^2(t)}dt$$

We will deal with this result after reducing the other part of the expression a bit. What we have is an error term which we know is bounded by some constant times $\frac{x}{\ln^2(x)}$. Since we really only want an asymptotic expression, we can attempt to bound the magnitude of this integral through a series of inequalities. The result of such bounding is the following:

$$\left|\frac{1}{n+1}\int_{1.5}^{x} t^{n-1}E(t)dt\right| \leq \frac{1}{n+1}\int_{1.5}^{x} t^{n-1}|E(t)|dt \leq C\int_{1.5}^{x} \frac{t^n}{\ln^2(t)}dt, \quad C \in \mathbb{R}$$

Now what we have is an integral from which we can create an error term. Since the integrand is increasing for all t greater than $e^{2/n}$ due to the fact that

$$\frac{d}{dx}\left(\frac{x^n}{\ln^2(x)}\right) = \frac{x^{n-1}(n\ln(x) - 2)}{\ln^3(x)}$$

is positive for all x greater than $e^{2/n}$, we can bound the increasing part of the integral by using the trivial estimation: assuming the maximum value on the interval and then multiplying by the length of the interval. The other part of the integral from 1.5 to $e^{2/n}$ either doesn't exist (in the case $e^{2/n} < 1.5$, the integral as it stands would be increasing for the entire interval) or is otherwise a constant for each particular n. The result is as follows:

$$C\int_{1.5}^{x} \frac{t^n}{\ln^2(t)}dt \leq D_1 + C\int_{e^{2/n}}^{x} \frac{x^n}{\ln^2(x)} \leq D_2 + C\frac{x^{n+1}}{\ln^2(x)}$$

Now $C\frac{x^{n+1}}{\ln^2(x)}$ is $\mathcal{O}\left(\frac{\pi(x^{n+1})}{\ln(x)}\right)$ because $\pi(x^{n+1}) \sim x^{n+1}/((n+1)\ln(x))$ and so we can rewrite our original equation as:

$$\sum_{p\leq x} p^n = \frac{x^{n+1}}{\ln(x)} - \frac{nx^{n+1}}{(n+1)\ln(x)} + \mathcal{O}\left(\frac{\pi(x^{n+1})}{\ln(x)}\right)$$

which simplifies to:

$$\sum_{p\leq x} p^n = \frac{x^{n+1}}{(n+1)\ln(x)} + \mathcal{O}\left(\frac{\pi(x^{n+1})}{\ln(x)}\right)$$

And since $\pi(x^{n+1}) = x^{n+1}/((n+1)ln(x)) + \mathcal{O}(\pi(x^{n+1})/((n+1)ln(x)))$ we obtain our desired result:

$$\sum_{p\leq x} p^n = \pi(x^{n+1}) + \mathcal{O}\left(\frac{\pi(x^{n+1})}{\ln(x)}\right)$$

Reversing the Formula

Suppose instead of an estimation of $\sum_{p\le x} p^k$, we would like an estimation of the prime counting function. We can get this simply by saying:

$$\pi(x) \approx \sum_{p\le x^{1/(k+1)}} p^k$$

(This is simply the main theorem with where n has been replaced with k and $x^{1/(k+1)}$ has been plugged in for x.)

So what does this even mean? Well, let us say, I wish to know the number of prime numbers less than 10^6. I don't really want to check every number up to 10^6 because that seems tedious. It is far easier to know every prime less than 10^3 or 10^2 so instead I make either of those lists. Say I make the 10^3 list. Now the theorem says that $\pi(10^6)$ is very nearly $\sum_{p\le 10^{6/(k+1)}} p^k$. Since I know the primes $\le 10^3$, I might as well use k=1. So the sum of primes less than 1000 is approximately the number of primes under 1000000. Neat. The natural question is to ask is "How approximate?" Lets find out!

$$\sum_{p\le 10^3} p = 76127$$

$$\pi(10^6) = 78498$$

Well, off by ≈ 2000. But give the formula some credit, that result is pretty close (For mathematically rigorous definitions of "pretty close").

This is particularly interesting to examine computationally because using computational methods, we can examine error in order to get a good idea of how close we are to $\pi(x)$ for different values of k and x by using the relative error. Relative error is good because it shows how far off we are, scaled to take into account the magnitude of the numbers.

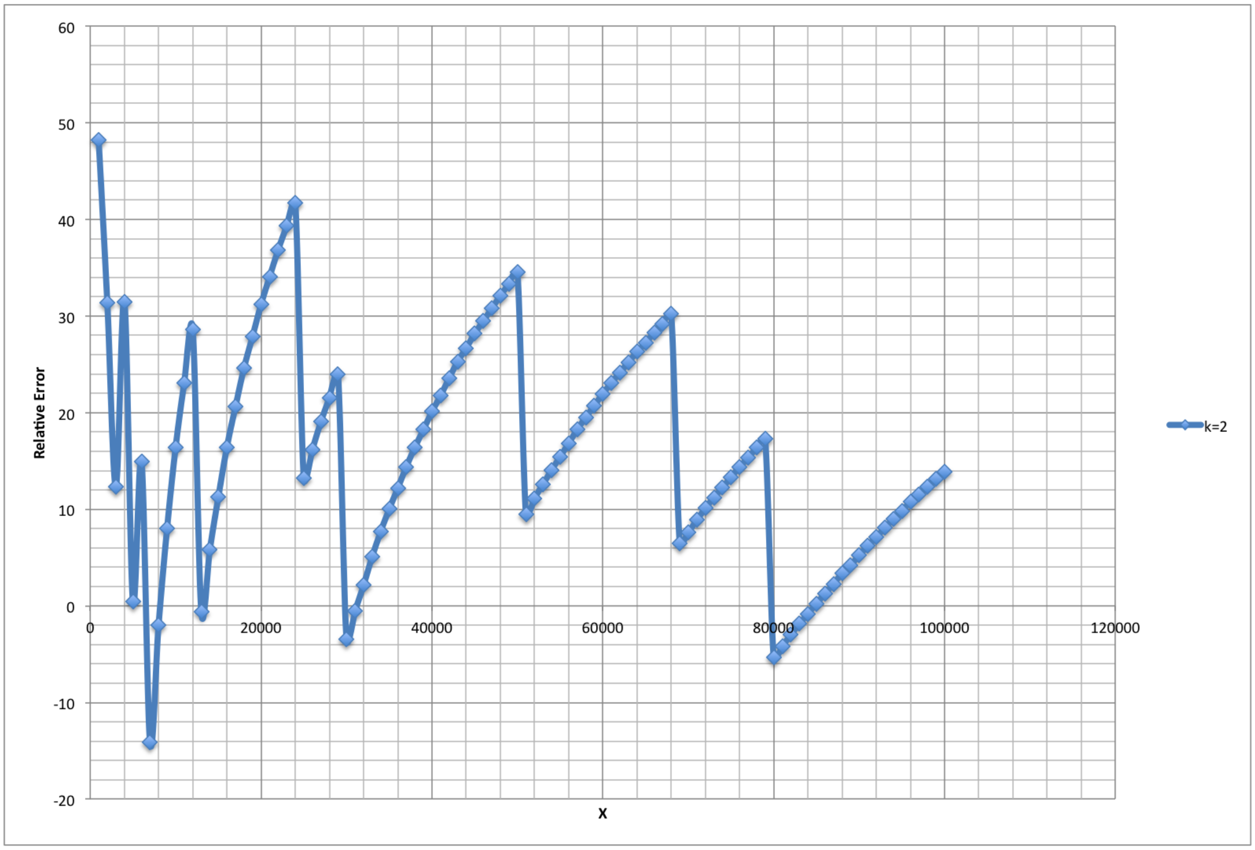

Figure 1: k=2 graph

This graph depicts the relative error for the k=2 case of our estimation function for value of x from 1000 to 100000. The relative error is

$$E(x) = 1 - \frac{\sum\limits_{p \leq x^{1/3}} p^2}{\pi(x)}$$

The graph in question reveals several important things about our function. The first of these is that it converges, something we already know from the asymptotic formula. The second thing that is noteworthy is the variation. Because we are taking the primes up to the cube root of x, we see a noticeable jump every time the cube root of x is prime. The error for x at this magnitude is too large to claim it is an accurate test. However, one thing to notice is that the error is sometimes negative. This is very different from the other graphs.

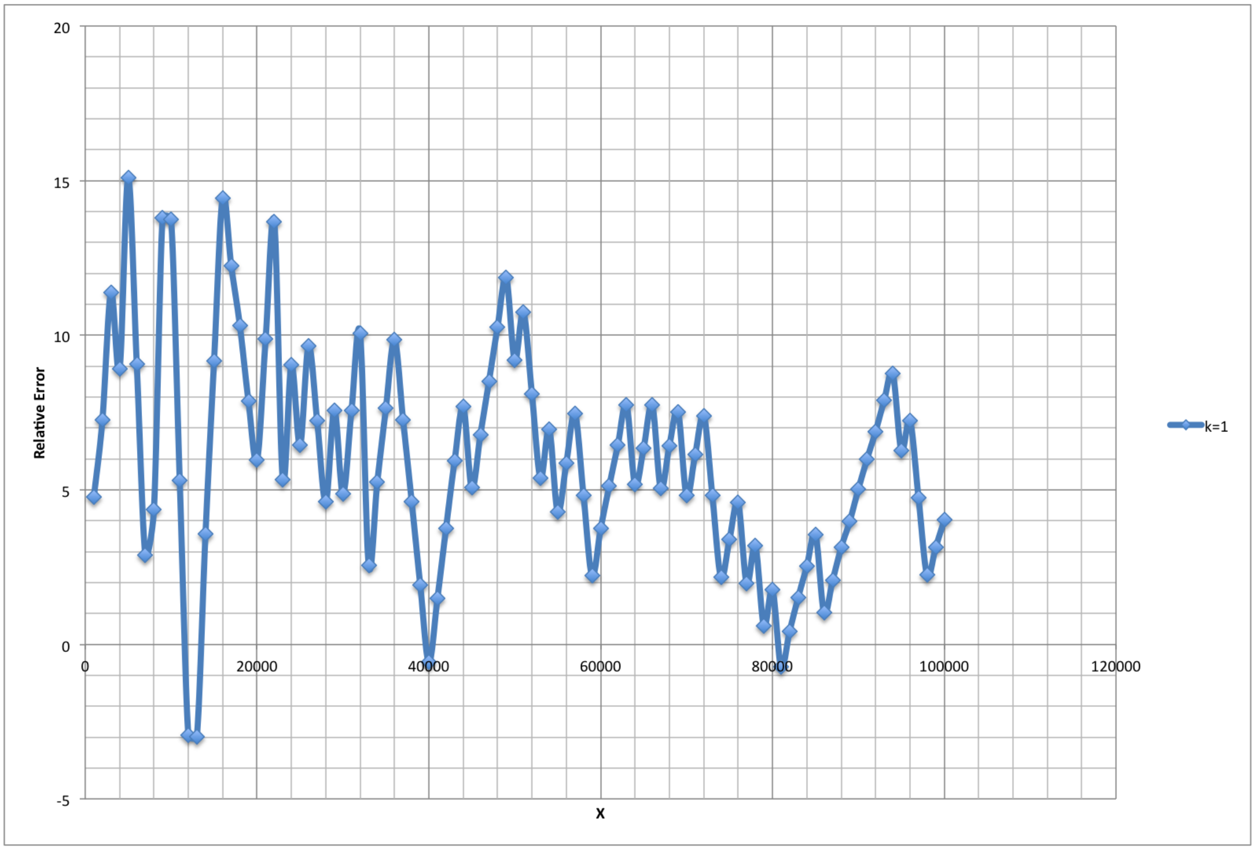

Figure 2: k=1 graph

This graph depicts the relative error for the k=1 case of our estimation function for value of x from 1000 to 100000. The relative error is

$$E(x) = 1 - \frac{\sum\limits_{p \leq x^{1/2}} p}{\pi(x)}$$

The graph again provides us with more insight. Once again we see that the graph is converging and this time in a much more pronounced way. The graph is erratic because the primes are occurring much more often. What's interesting to note is the fact that we see it go negative in similar places. This could be a coincidence, but it's doubtful. It is likely that at these points prime distribution is somewhat irregular for either the prime counting function or the prime power sum. This is, however, merely a conjecture.

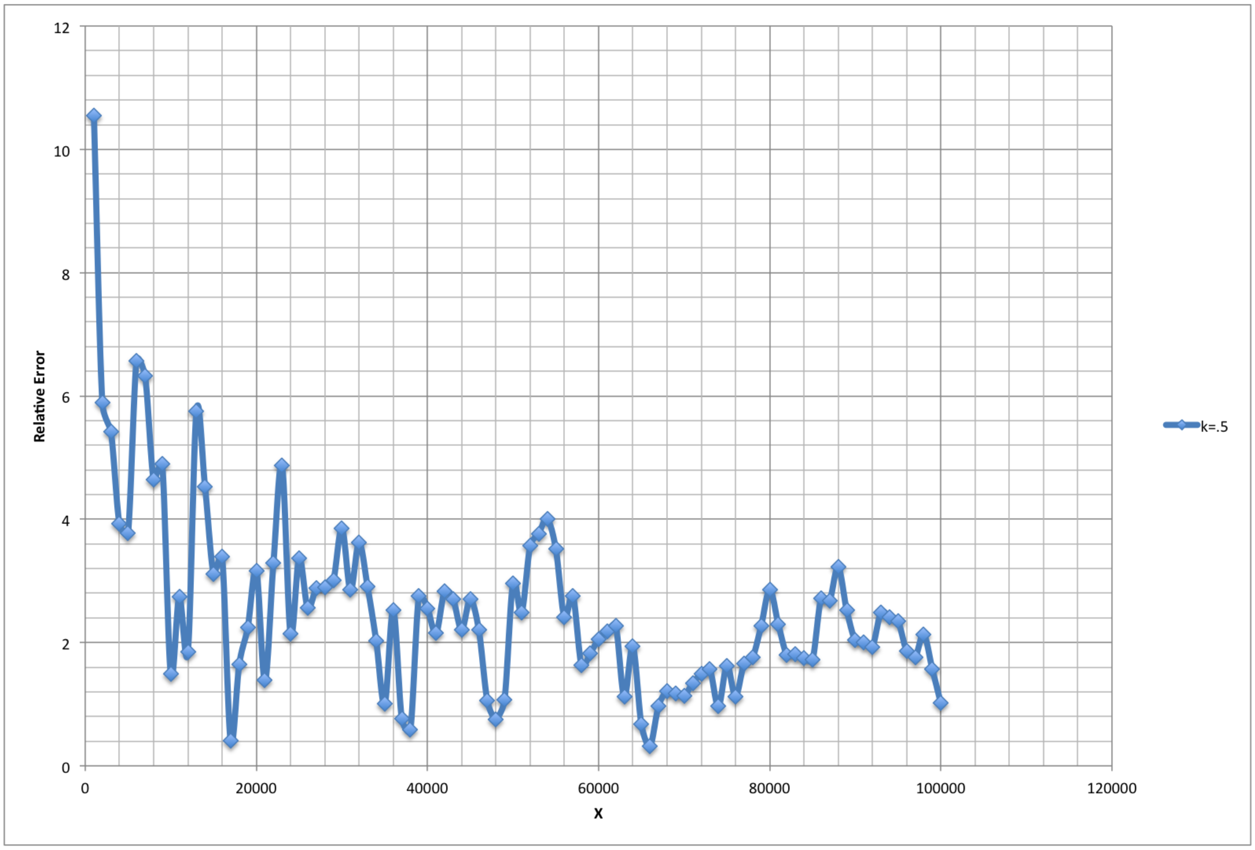

Figure 3: k=.5 graph

This graph depicts the relative error for the k=1/2 case of our estimation function for value of x from 1000 to 100000. The relative error is

$$E(x) = 1 - \frac{\sum\limits_{p \leq x^{2/3}} p^{1/2}}{\pi(x)}$$

Here we can see that the error is becoming quite small - only 2% at x=10^5. The convergence is now extremely clear and interestingly enough, we have our first graph with no negative relative errors, which means that there may not be any. It would be nice to have a proof either confirming or denying this hypothesis.

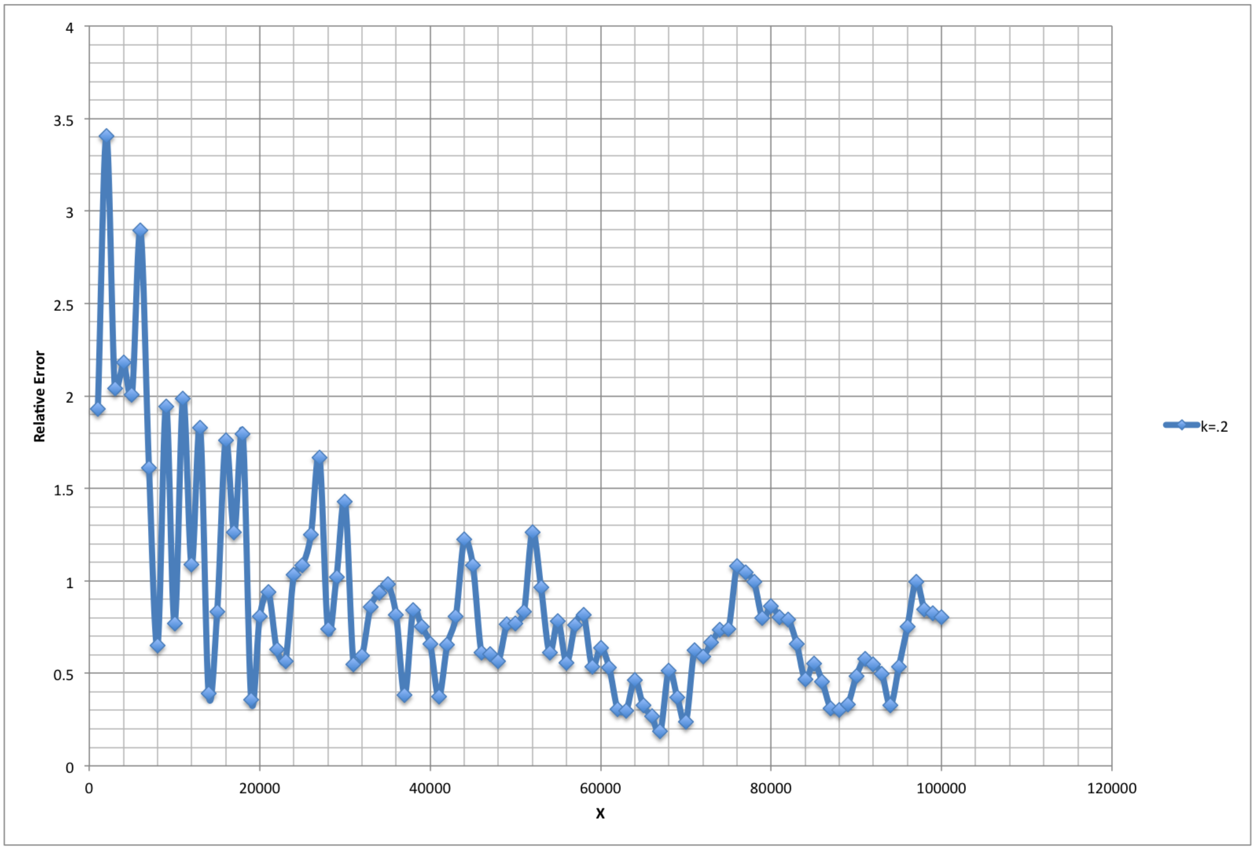

Figure 4: k=.2 graph

This graph depicts the relative error for the k=1/5 case of our estimation function for value of x from 1000 to 100000. The relative error is

$$E(x) = 1 - \frac{\sum\limits_{p \leq x^{5/6}} p^{1/5}}{\pi(x)}$$

The final graph that I have decided to include is the best at estimating the error. The reason is simple. The fifth root of the largest prime is less than 10 and we're only counting up to $10^{5/6}$ and as we let our k approach 0 the error must also approach 0. To better understand this, at k=0 the Prime Power sum literally adds 1 for every prime up to x which is exactly prime counting function. Also no 0 crossings here either, so there's still a lot to be uncovered surrounding that issue.

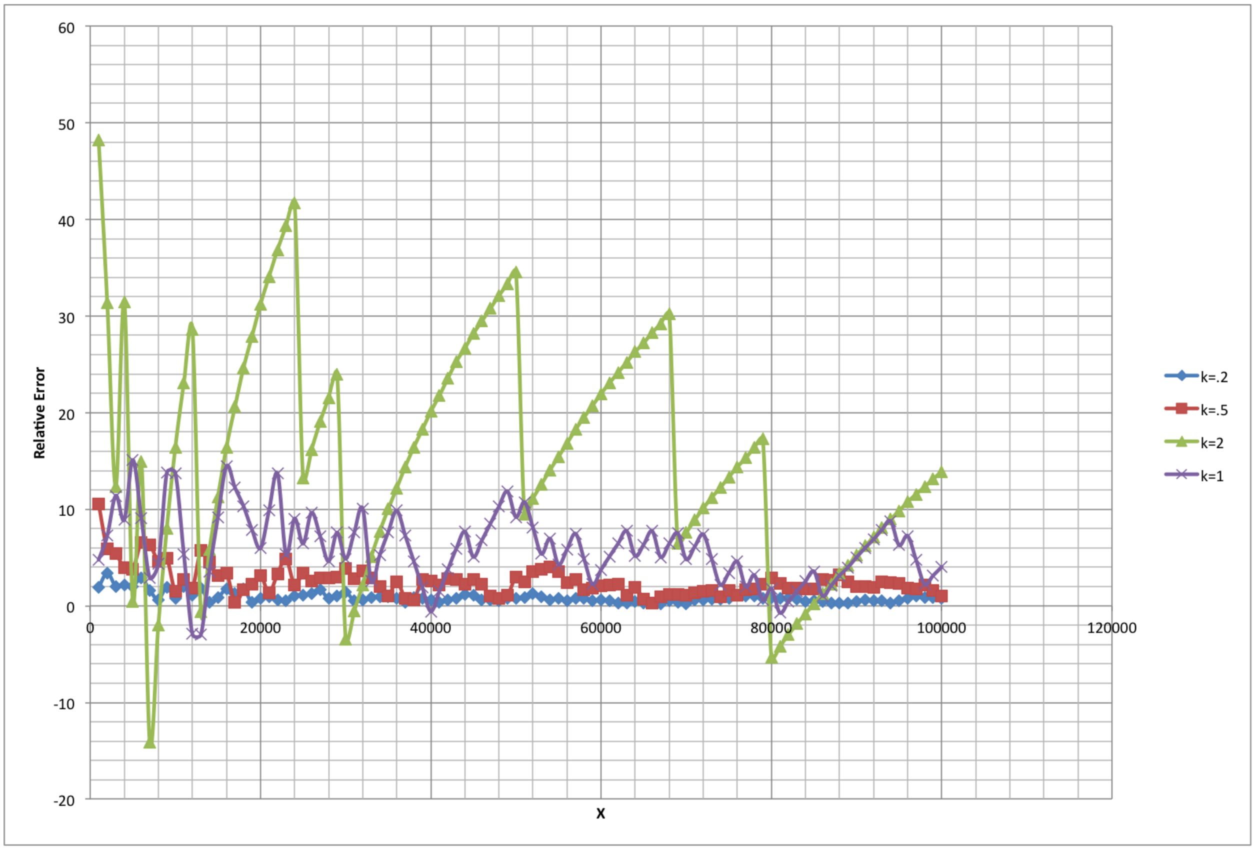

Figure 5: Overlay Graph

This is simply the same graphs overlapped to give insight and perspective as to how k affects how these graphs behave. What is important to note is the reduction of variation in the error as k becomes small, because as mentioned before, as k goes to 0, E(x) goes to 0. This makes sense, at least from the standpoint of intuition, because as k decreases, our formula sums more primes. And common sense says the more we know about the primes before, the better we can predict the primes afterwards. That was, of course, the fascinating result of our research and it is illustrated clearly in our graphs.

Further Research

There are several possibilities for further research that this research presents. One is determining the question of whether or not the relative error function crosses 0 infinitely often and if so for what values of k? One observation which we made was that there tends to be a bias. The error is almost always positive and rarely falls below 0. Another possibility is to find a more accurate error bound. There is some possibility that the solution to this question lies in an understanding of explicit formulae for the Riemann Zeta Function. Another possible path for research could be to investigate whether or not this holds true for the modular case. For instance, does the sum of primes which are 3 mod 4 and less than x predict the number of primes 3 mod 4 and less than x^2? These are all questions which could certainly produce some intriguing results.

References

Bach, E. (1996). Algorithmic number theory: Efficient algorithms.

Ingham, A. E. (1990). The distribution of prime numbers. Cambridge: Cambridge University Press.

Microsoft. (2011). Microsoft Excel [computer software]. Redmond, Washington: Microsoft.

Pari/GP version 2.7.2, The Pari Group

Riemann–Stieltjes integral. (2014, August 7). In Wikipedia, The Free Encyclopedia. Retrieved 14:10, November 12, 2014, from http://en.wikipedia.org/w/index.php?title=Riemann %E2%80%93Stieltjes_integraloldid=620208655

∼Original paper completed on October 25^{th} 2014. Some changes were made for this submission.