ON THE CONSTRUCTIBILITY OF n-DIVISION POINTS OF CERTAIN POLAR CURVES BY AREA

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1. Abstract

In this paper, we develop a method for a problem not heavily touched upon by existing mathematical literature: evaluating the constructibility of n-division points by area of any given closed polar curve, with n-divisions defined as intersections of a closed curve with rays starting at the origin that divide the curve into n sectors of equal area. Positive results were found for the ellipse, the inner loop of the Maclaurin trisectrix, and the k-petaled lemniscate. Our method involved finding a closed form solution that expressed the radius of the curve in terms of the traversed area in order to analyze whether certain divisions were constructibly possible. When this expression was algebraic and constructible, using this method yielded the following results: the n-divisions of an ellipse are constructible if n is constructive for a circle, and the n-division points of the inner loop of the Maclaurin trisectrix and of the k-petaled lemniscate are constructible for all positive integers n. Though the results for the ellipse have already been proven by Kepler, the results for the Maclaurin trisectrix and the k-petaled lemniscate are original. Furthermore, our method suggests that other curves with transcendental expressions for radius in terms of area do not have constructible n-division points. A secondary result of our work is that the constructible n-division points are also constructible without the given curve for the ellipse and the inner loop of the Maclaurin trisectrix, though not for most k-lemniscates.

2. Introduction

Constructing and n-dividing different polar curves through the use of straightedge and compass constructions are proven using Field Theory. Some of the solved systems include arc length divisions for circles, hypocycloids, and lemniscates. However, little has been done in terms of n-dividing area of pre-drawn polar curves. Using field theory allows for a closed determination of the possible n-divisions, since it gives a closed form for possible lengths and angles. Constructibility can be applied to the long-standing ancient Greek "unsolvable" problems, roots of unity in complex analysis, and computer science through binary digits. Furthermore, the Maclaurin trisectrix has applications in the geometric problems of antiquity, specifically angle trisection, hence the title of the curve.

Date: April 7, 2015.

3. Definitions

Definition 1 (Polar Curve)**.** Let r be the distance from the origin and θ be the angle in the counter-clockwise direction from the positive x-axis. If r can be written as a function of θ, then the graph of r is called a *polar curve*.

Definition 2 (n-Divisions of a Polar Curve by Length)**.** A set of n points on the curve enumerated in the counterclockwise direction $\{1, 2, 3, ..., n\}$ is said to *n-Divide a Polar Curve by length* if the points divide the curve into n arcs of equal length.
Note: n-divisions by length are well-defined only for closed curves.

Definition 3 (n-Divisions of a Polar Curve by Area)**.** A set of n points on the curve enumerated in the counterclockwise direction $\{1, 2, 3, ..., n\}$ is said to *n-Divide a Polar Curve by Area* if the points divide the curve into n sectors of equal area.
Note: n-divisions by area are well-defined only for closed curves.

Definition 4 (k-Petaled Lemniscate)**.** A *k-petaled lemniscate* is a curve of the form $r^2(\theta) = \cos(k\theta)$, where k is an integer.

Definition 5 (Group)**.** A *group* is a set of elements, G, that is closed under an associative binary operation, $*$, that also satisfies the following two properties:

(i) There exists an identity element $e \in G$ such that $e * g = g$ for all $g \in G$.
(ii) Every element $g \in G$ has an inverse g^{-1} such that $g^{-1} * g = g * g^{-1} = e$.

Definition 6 (Field)**.** A *field* is a set of elements, F, which is closed under two commutative binary operations, $*$ and $+$, such that $F \setminus \{0\}$ is a group under multiplication, $*$, and F is a group under addition, $+$.

Definition 7 (Field Extension)**.** If L is a field and K is a subfield of L, denoted $K \leq L$, then L is called a *Field Extension* of K.
Furthermore, $L = K[\alpha_1, \alpha_2, \ldots, \alpha_n]$ can be expressed as a vector space over K with basis vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$.
The dimension of L as a vector space over K, n is denoted $[L : K]$ and is called the degree of the extension.

Definition 8 (Fermat Prime)**.** A *Fermat Prime* is a prime number, $p = 2^{2^t} + 1$ with $t \in \mathbb{N}$.

4. Constructibility

Constructibility originally began as a Classical Greek geometry problem and focuses on what can be created solely with a straightedge and compass. While the Greeks could only pose difficult problems and solve them, the use of Field Theory and Galois Theory can both prove difficult constructions as well as disprove the existence of others.

4.1. **Addition and Subtraction.** If m and n are constructible, then adding or subtracting parallel lines of length m and n will yield a line segment with the desired length.

4.2. **Multiplication and Division.** To construct the length mn, one can first construct the points $(0, m)$ and $(n, 0)$. Then one should construct a line l_1 from $(0, 1)$ to $(n, 0)$. Lastly, since the intersection of l_2 and the x-axis will be $(mn, 0)$, constructing a line l_2 parallel to l_1 that passes through $(0, m)$ completes the problem.

4.3. **Square Root.** The figure below illustrates the procedure used to construct a line segment of length $\sqrt{m}$ given a line segment of length $m + 1$.
Drawing a circle whose diameter is the line segment $\overline{(-1, 0), (m, 0)}$, would intersect the y-axis at $(0, \sqrt{m})$ as shown.
This can be easily proven using similar triangles.

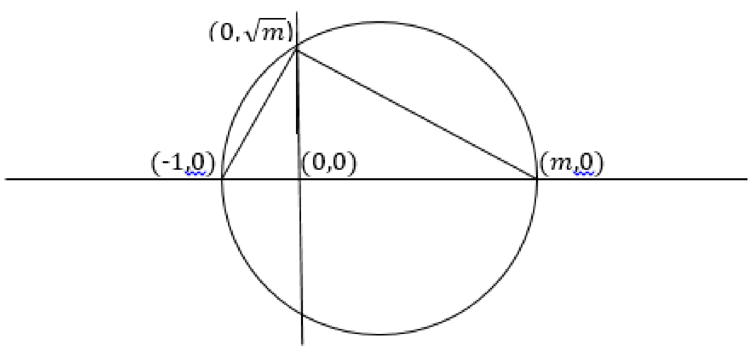

Figure 1. Constructing $\sqrt{m}$

4.4. **The Results in Terms of Field Theory.** Since our two available tools are a straightedge and a compass, points can be constructed from three different types of intersections: straightedge/straightedge, straightedge/compass, and compass/compass. The first is the intersection of two lines, meaning that the construction will always stay in the field, since the intersection of two lines yields a linear equation. The intersection of a circle and a line is an intersection of a quadratic and a linear system, which can then be reduced into a quadratic equation. This reduction to a quadratic equation applies similarly to the intersection of two circles, which represents the intersection of two quadratic systems. Thus, for any intersection point (x, y), regardless of the type, x and y can be reduced down to a quadratic equation in terms of the elements of the current field. This means that, for any element α constructed from a field $K \subset \mathbb{R}$ it must be true that $p\alpha^2 + q\alpha + r = 0$, for some p, q and $r \in K$. This would imply that

$$\alpha^2 = \frac{-q\alpha - r}{p}$$

meaning that any element in $K(\alpha)$ can be written in the form $c + d\alpha$ where $c, d \in K$.

Thus any intersection either preserves the field, or imposes a degree-2 extension on the field. Furthermore, because $\alpha = \dfrac{-q \pm \sqrt{q^2 - 4pr}}{2p}$, any element generated from a degree-2 extension is generated from a combination of the five constructible operations discussed in the previous section.

Therefore, any value that can be created from a finite chain of degree-2 extensions over the $\mathbb{Q}$ is a constructible number. Furthermore, the converse is true that any value that is outside the reach of any chain of degree-2 extensions is not constructible. In other words, a number is constructible if and only if it can be created through a series of the constructible operations which embody all degree-2 extensions: addition, subtraction, multiplication, division, and square root.

5. Previous Theorems on Arc Length

Theorem 1 (Gauss-Wantzel)**.** *A regular n-gon is constructible if and only if it has n sides where $n = 2^q p_1 p_2 p_3 \cdots p_j$ such that $q \in \mathbb{N}$ and each of the p_i's is a distinct Fermat prime. Thus, all constructible angles must be of the form $\dfrac{2\pi k}{n}$, where n is of the above form, and $k \in \mathbb{Z}$.*

Theorem 2 (Abel)**.** *A lemniscate, a curve of the form $(x^2 + y^2)^2 = c(x^2 - y^2)$ or $r^2 = \cos(2\theta)$, is n-divisible by length for the same values as a circle which is when $n = 2^q p_1 p_2 p_3 \cdots p_j$ such that $q \in \mathbb{N}$ and each of the p_i's is a distinct Fermat prime.*

Theorem 3 (Mani-Salzedo)**.** *A hypocycloid is a curve traced out by the trajectory of a fixed point, P, on a circle of radius 1 that rolls around the circumference of a circle of radius $c > 1$. For all pre-drawn hypocycloids, the n-divisions for arc length are constructible for all natural numbers n.*

Thus, the question of the constructibility of n-divisions by length have been solved and proven for the circle, the 2-lemniscate, and all hypocycloids. However, as discussed in the Introduction, little has been done in the pursuit of n-divisions by area, which is the topic we will explore in the upcoming sections. Physicist Johaness Kepler did analyze n-divisions by area of the ellipse, though there is little literature on area divisions for other polar curves. Furthermore, note that, for a circle, the n-division points by area are equal to the n-division points by length.

6. Our Method

It is known that we can find a function of the area A in terms of the angle traversed via the polar area formula:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

For certain curves, this allows us to define a function of φ, the angle relative to the positive x-axis, in terms of A, the area traversed by angle φ. Using the polar curve equation $r(\varphi)$, we can then define a function $r = \rho(A)$. Finally, if the radius at each nth division of the total area A_0 is of constructible length, i.e. if $\rho(\frac{m}{n}A_0)$ is constructible for all $1 \leq m < n$, then the mth division point can be attained by drawing a circle of radius $\rho(\frac{m}{n}A_0)$ centered at the origin, implying all n-divisions are constructible for that value of n. Furthermore, note that this method requires the curve to be drawn in the first place in order for its division points to be constructed. However, by evaluating both r and θ, we can evaluate whether division points using the method are constructible without the given curve.

7. The Ellipse

Without loss of generality, since the ratio of the major axis to the minor axis is what distinguishes ellipses from one another, we can restrict our selection of ellipses to those with a vertical minor axis of length 2 and a horizontal major axis of $2a$, for some $a \in \mathbb{R}$. We can, then, write the general rectangular coordinate equation as

$$\frac{x^2}{a^2} + y^2 = 1$$

Kepler, using the following analysis unique to ellipses, proved that the n-divisions of circles directly correspond to n-divisions of ellipses. Using the rectangular coordinate equation, we get that $y = \pm\frac{1}{a}\sqrt{a^2 - x^2}$, which for all x-coordinates is exactly $\frac{1}{a}$ times the height of the circle of radius a centered at the origin. Thus, the area traversed by the circle over any x-interval will also maintain this ratio and be a times the area traversed by the ellipse over the same interval. After some algebra, one can quickly prove that the n-division points by area of a circle will have equal x-coordinates as the n-division points of an ellipse. Thus, as the y-coordinates differ only by the ratio of a, the n-division points for an ellipse are constructible if and only if they are constructible for a circle. Our method, though less straightforward and elegant, provides an alternative proof for the same conclusion reached by Kepler. Though the method may be less effective on the already solved problem of the ellipse, its use in this case allows us to verify the results of the method and apply it effectively for other curves. We start our method by substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into the rectangular equation, which yields $r^2(\cos^2(\theta) + a^2\sin^2(\theta)) = a^2$. This gives the following polar equation:

$$r(\varphi) = \frac{a}{\sqrt{a^2 + (1 - a^2)\cos^2(\varphi)}}.$$

Thus, the area

$$\begin{aligned}A(\varphi) &= \frac{1}{2}\int_0^{\varphi} r^2 d\theta \\ &= \frac{1}{2}\int_0^{\varphi} \frac{a^2}{a^2+(1-a^2)\cos^2(\varphi)} d\theta \\ &= \frac{a}{2}\arctan(a\tan(x)).\end{aligned}$$

However, since A is a periodic function with period π, this expression for $A(\varphi)$ is accurate only for $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. This gives the total ellipse area

$$\begin{aligned}A_0 &= 4\lim_{x\to\frac{\pi}{2}} A(x) \\ &= 2a\lim_{x\to\frac{\pi}{2}} \arctan(a\tan(x)) \\ &= \pi a.\end{aligned}$$

Solving for φ in terms of A yields

$$\varphi = \arctan(\frac{1}{a}\tan(\frac{2A}{a})).$$

Substituting this expression for φ into $r(\varphi)$ gives us

$$\rho(A) = \frac{a}{\sqrt{a^2+(1-a^2)\cos^2(\arctan(\frac{1}{a}\tan(\frac{2A}{a})))}}.$$

Since $\cos(\arctan(u)) = \pm\frac{1}{1+u^2}$, then

$$\begin{aligned}\rho(A) &= \frac{a}{\sqrt{a^2+\frac{1-a^2}{1+(\frac{1}{a}\tan(\frac{2A}{a}))^2}}} \\ &= \frac{1}{\sqrt{1+\frac{1-a^2}{a^2+\tan^2(\frac{2A}{a})}}} \\ &= \sqrt{\frac{a^2+\tan^2(\frac{2A}{a})}{1+\tan^2(\frac{2A}{a})}}\end{aligned}$$

Since $1+\tan^2(\frac{2A}{a}) = \sec^2(\frac{2A}{a})$, we can continue this simplification.

$$\begin{aligned}\rho(A) &= \sqrt{\cos^2(\frac{2A}{a})[a^2+\tan^2(\frac{2A}{a})]} \\ &= \sqrt{a^2\cos^2(\frac{2A}{a})+\sin^2(\frac{2A}{a})}.\end{aligned}$$

Thus the radius for the m^{th} n-division of A_0 is equal to

$$\rho(A_0\frac{m}{n}) = \rho(\frac{\pi a m}{n}) = \sqrt{a^2\cos^2(\frac{2\pi m}{n})+\sin^2(\frac{2\pi m}{n})}.$$

This corresponds to the point $(a\cos(\frac{2\pi m}{n}), \sin(\frac{2\pi m}{n}))$ which aligns exactly with Kepler's result. Thus, the point is constructible if and only if $\cos(\frac{2\pi m}{n})$ is constructible, which is true only for constructive n values: $n = 2^q p_1 p_2 p_3 \cdots p_j$ where $q \in \mathbb{N}$, p_i's are distinct Fermat primes. Furthermore, the n-division point, when constructible, is constructible without the given curve, since each of the x and y coordinates are constructible individually.

8. The MacLaurin Trisectrix

The polar equation for the Maclaurin trisectrix is $r = \sec(\frac{\phi}{3})$, for $-\frac{3\pi}{2} \leq \varphi \leq \frac{3\pi}{2}$. However, the inner loop is bounded by angles $-\pi$ and π. Thus, for $-\pi \leq \pi$, the angle traversed by the inner loop from $-\pi$ to φ is given by

$$\begin{aligned} A(\varphi) &= \frac{1}{2}\int_{-\pi}^{\varphi} r^2 d\theta \\ &= \frac{1}{2}\int_{-\pi}^{\varphi} \sec^2(\frac{\theta}{3}) d\theta \\ &= \frac{3}{2}[\tan(\frac{\varphi}{3}) + \sqrt{3}] \end{aligned}$$

As a result, this gives us a total inner loop area of $A(\pi) = 3\sqrt{3}$. Additionally, we have $\varphi = 3\arctan(\frac{2A}{3} - \sqrt{3})$ when solving for ϕ in terms of A. Substituting this expression into $r(\varphi)$ yields

$$\begin{aligned} \rho(A) &= \sec(\arctan(\frac{2A}{3} - \sqrt{3})) \\ &= \sqrt{1 + (\frac{2A}{3} - \sqrt{3})^2} \\ &= \frac{2}{3}\sqrt{A^2 - 3\sqrt{3}A + 9}. \end{aligned}$$

Thus, since $\dfrac{m}{n} \in \mathbb{Q}$ is always constructible, the radius of the m^{th} n-division of the inner loop area

$$\rho(3\sqrt{3}\frac{m}{n}) = 2\sqrt{\frac{3m}{n}(1 - \frac{m}{n}) + 1}$$

is n-divisible by area, for all $n \in \mathbb{N}$.

Furthermore, note that solving φ in terms of r yields $\varphi = 3\text{arcsec}(r)$. Additionally, if r is a constructible value, then the angle $\text{arcsec}(r)$ is also constructible,

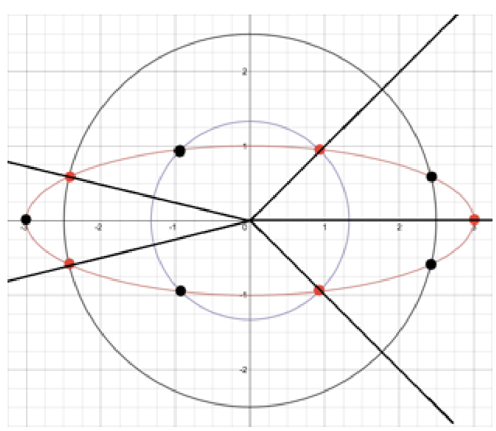

Figure 2. An ellipse with $a = 3$ divided into 5 sectors of equal area

for a right triangle with a hypotenuse of length r and a leg of length 1 can then easily be constructed. Thus, each division point of the inner loop of the Maclaurin trisectrix can be constructed without a given loop.

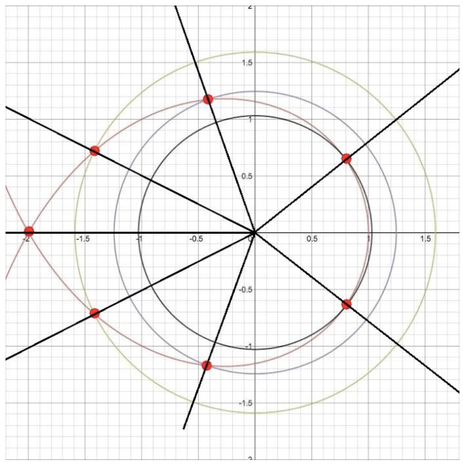

FIGURE 3. The inner loop of the Maclaurin trisectrix dissected into 7 sectors of equal area.

9. The k-Petaled Lemniscate

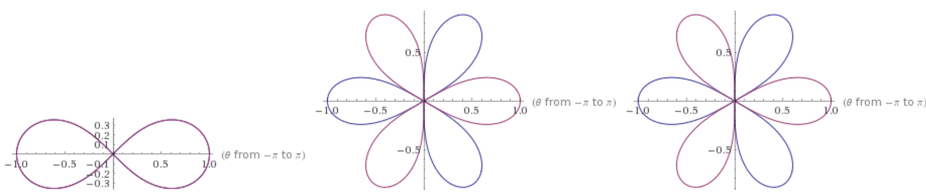

FIGURE 4. The 2,3, and 4-Petaled Lemniscates, from left to right

The k-petaled lemniscate has a polar equation of $r^2 = \cos(k\varphi)$, where $k \in \mathbb{Z}$. Thus, since each petal spans an angle of $\frac{\pi}{k}$, the area A of the first petal is

$$\begin{aligned} A(\varphi) &= \frac{1}{2}\int_{-\frac{\pi}{2k}}^{\varphi} r^2 d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{2k}}^{\varphi} \cos(k\theta) d\theta \\ &= \frac{1}{2}[\sin(k\varphi) + 1] \end{aligned}$$

where $-\frac{\pi}{2k} < \varphi \leq \frac{\pi}{2k}$. This gives a total area of $A(\frac{\pi}{2k}) = \frac{1}{k}$ for each petal of k-petaled lemniscate. Now that we have $A = \frac{1}{2k}[\sin(k\varphi)+1]$, solving for φ yields

$\varphi = \dfrac{1}{k}\arcsin(2kA-1)$. Substituting this expression for φ into $r = \sqrt{\cos(k\varphi)}$ yields

$$\begin{aligned}\rho(A) &= \sqrt{\cos(\arcsin(2kA))}\\ &= \sqrt{1-(2kA-1)^2}\\ &= 2\sqrt{kA(1-kA)}\end{aligned}$$

where $0 \leq A < \frac{1}{k}$. For the total area of the k-lemniscate, there are two cases to be considered. If k is an odd value, then lemniscate has $2k$ petals, so the total area $A_0 = 2$. If k is an even value, then the lemniscate has k petals, so the total area $A_0 = 1$.

In general, the m^{th} n-division of the total area is $\dfrac{mA_0}{n}$. However, ρ is only well-defined for $0 \leq A < \frac{1}{k}$, i.e., for a single petal. Thus, one can pursue the following procedure to find the n-division points of a k-lemniscate. One can pursue the following procedure to find the n-division points of a k-lemniscate. One can find the m^{th} and $(n-m)^{th}$ division points of each petal by drawing a circle of radius $\rho(\frac{m}{kn})$, for $m \leq \frac{n}{2}$, since

$$\rho(\frac{m}{kn}) = \rho(\frac{1}{k} - \frac{m}{kn}) = \rho(\frac{n-m}{kn}).$$

This radius is constructible because $\frac{m}{n} \in \mathbb{Q}$ and $\rho(\frac{m}{kn}) = 2\sqrt{\frac{m}{n}(1-\frac{m}{n})}$. Thus, this procedure constructibly divides each petal into n sectors. Furthermore, note how the values of ρ are independent of the value of k. For an odd k, the entire k-lemniscate is then divided into $2kn$ sectors, so selecting every $2k^{th}$ point gives n equal sectors. Similarly, for an even k, the entire k-lemniscate is divided into kn sectors, so selecting every k^{th} point gives n equal sectors. Thus, the n-division points of a k-lemniscate are constructible for all $n, k \in \mathbb{N}$.

Now, note that solving for φ in terms of r yields

$$\varphi = \frac{1}{k}(\arccos(r^2) + \pi p)$$

where $p \in \mathbb{Z}$. Thus the angle of a division point of radius

$$\rho(\frac{m}{kn}) = \frac{1}{k}(\arccos(\frac{4m}{n}(1-\frac{m}{n}) + \pi p)).$$

The k^{th} Chebyshev polynomial of the first kind is

$$\begin{aligned}\cos(kx) &= T_k(\cos(x))\\ &= \frac{e^{ikx}+e^{-ikx}}{2}\\ &= \frac{(\cos(x)+i\sin(x))^k + (\cos(x)-i\sin(x))^k}{2}\\ &= \sum_{i=0}^{k}\binom{k}{i}\cos^i(x)\sin^{k-i}(x)\cos(\frac{1}{2}(k-i)\pi)\\ &= \sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{k}{2i}\cos^{k-2i}(x)(1-\cos^2(x))^i\end{aligned}$$

which is a k^{th} degree polynomial with integer coefficients. If $T_k = \sum_{i=0}^{k} a_i x^i$, then where θ is the angle of some division point,

$$\begin{aligned} T_k(\cos(\theta)) &= \cos(k\theta) \\ &= \frac{4m}{n}(1 - \frac{m}{n}) \\ &= \sum_{i=0}^{k} a_i \cos^i(\theta) \end{aligned}$$

then $\cos(\theta)$ is a root of the k^{th}-degree polynomial $T_k(x) - \dfrac{4}{m}(1 - \frac{m}{n})$, which has all rational coefficients. In other words, $\cos(\theta)$ is an element of a k^{th} degree extension of $\mathbb{Q}$. Thus, when $k = 2^i$ for some integer i, all n-division points of the k-lemniscate are constructible without the given curve. On the other hand, when k is not a power of 2, the n-division points of the k-lemniscate are not all constructible without the given curve.

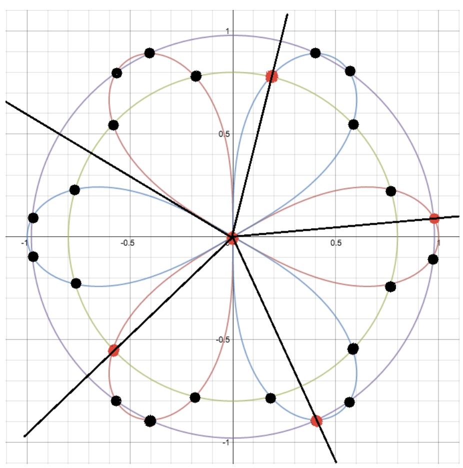

FIGURE 5. A 3-lemniscate divided into 5 equal sectors

10. Results

Kepler used simple geometry to prove that the conditions for constructibility of a circular division apply also to an ellipse. Our method returned the same results and provided an alternative proof to Kepler's conclusion. Furthermore, our method can be used more generally to evaluate n-divisibility for other curves. We proved that the Maclaurin trisectrix, which had not been n-divided in the past, can be divided into n equal sectors by area for any integer n. The curve was previously used in constructibility by Maclaurin to prove the impossibility of arbitrary angle trisection. Two hundred years ago, Niels Abel proved that the n-division points of a lemniscate were constructible for n of the form $2^q p_1 p_2 * \cdots p_n$, where each p_i is a distinct Fermat Primes and q is a natural number. He never, however, considered the family of curves of the form $r^2 = \cos(k\varphi)$. Since this family of curves has not been studied before, we decided to name this family the k-petaled lemniscates. Furthering Abels research, in this paper, we have proven that the n-division by area of any pre-drawn k-petaled lemniscate is constructible for all integers n. Lastly, for each curve, we determined whether the n-division points would be constructible without the pre-drawn curve. For the ellipse and the Maclaurin trisectrix, all n-division points are constructible without the pre-drawn curve. For the k-petaled lemniscate, all n-division points are constructible without the given curve only when k is a power of 2.

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