Odd Dunkl Operators and nilHecke Algebras

Ritesh Ragavender

1 Process

I've always been interested in mathematics. It is the pinnacle of human logic and is unquestionably correct, leading to wonderful models of predicting weather and making transistors. I found math to be a beautiful art form with a personality; some equations are humble, some are lawless, and some are mysterious, teasing for further inquiry. I started getting pretty involved in math in freshman year, becoming a squad leader of the championship Lehigh Valley ARML Fire team for the past four years, becoming an individual finalist at the Princeton University Math Competition, and qualifying for the USA Math Olympiad. But there came a point when I was a little disappointed by the fact that all the IMO problems that I had solved had already been solved. I wanted to do something new. I also wanted to give back to a society that had nurtured me so much and introduced me to the wonderful world of mathematics. Doing math research was the best way for me to do this, because the social benefits and the personal happiness of mathematics makes it worthwhile.

I knew that I wanted to do a project in modern algebra, which is exactly what I did after gaining acceptance into MIT PRIMES USA, my research program in mathematics. This process was very time consuming, and I typically invested at least fifteen hours a week into research. I initially focused on learning the background material in representation theory, my mentor's work, and in Stanley's *Enumerative Combinatorics*, a treasure trove of algebraic combinatorics. I started this project in mid December of 2012, and I'm still continuing it, trying to prove some of my remaining conjectures.

I began my work work by learning the background material, which became necessary in order to understand proofs even later in the research period. I first tried to study q-symmetric polynomials, a general construct of what my mentor was studying. Later, I generalized certain diagrammatic methods to find relations between elementary q-symmetric polynomial in this case.

Doing a project in a very high-level topic can definitely be frustrating and difficult at times. There were many weeks where I could hardly make any progress on a certain problem, even though I tried to solve the problem every day. Some of these problems (for example, the problem of completely classifying relations when $q^3 = 1$) are still unresolved. When I couldn't solve a problem, I consulted my mentor for problem solving strategies and further advice. I attempted different techniques when I could not arrive at a solution, and kept track of what worked and what did not. I also tried, like Richard Feynman and John Conway, to have a few problems at the back of my mind at all times in order to work on. Thus, by balancing work between smaller, more feasible problems, as well as harder problems, I could make progress more regularly. I learned how to keep my eye on the big picture of oddification, and to take larger strides after internalizing certain concepts (for example, to work out the case of general n after working out $n = 2$ and 3). Sometimes, I had to write out my calculations on a poster board because paper did not offer enough horizontal space. Reading papers was also a very challenging part of research, which I gradually got better at by trying my best to make the notation intrinsic. I can now help edit my mentor's new work, which makes me feel very accomplished!

My research has given me some wonderful traveling opportunities as well. I've traveled to China to present my research in the Shing-Tung Yau Math Research contest, where I also saw the Great Wall and the Summer Palace. I also had the opportunity to go to Los Angeles for the Intel ISEF, which was an experience filled with fun times with new friends. I got to experience many cultural and social aspects of the world through my scribbling and typing at home!

I definitely recommend to anyone interested in math and science to take initiative and find others willing to help you. Find a problem that interests you, for personal or scientific reasons, and pursue it. Tell all your teachers about it, and find people as passionate about it as you are. Doing research and contributing to the interface of math and science feels amazing!

2 Outline of Research

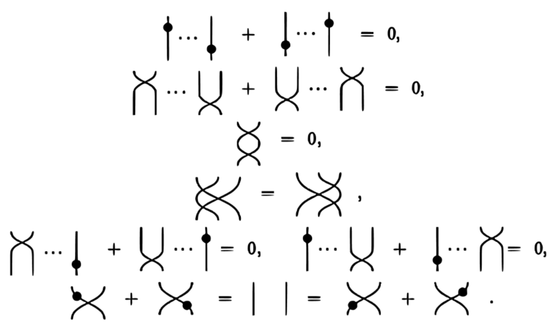
I have conducted research in representation theory, the backbone of many mathematical ideas in algebra, topology, and particle physics. A major part of this field is the interplay between symmetries and the algebraic objects which control them. In the 1980's, Charles Dunkl introduced certain operations involving both derivatives (rates of change) and certain reflections naturally associated to the symmetry of ordinary Euclidean space. These Dunkl operators have proven useful in both physics and mathematics, where they are used to study quantum many-body problems, conformal field theory, Lie theory, and harmonic analysis. In my project, I studied new Dunkl-type operators better adapted to a type of noncommutative space, which is a space in which the multiplication of quantities does not satisfy the familiar relation $ab = ba$.

Noncommutativity may seem counterintuitive, but it appears frequently in mathematics and the study of quantum mechanics. For example, Heisenberg's famous uncertainty principle involves the noncommutativity of the position and momentum operators. Matrix multiplication and the cross product, both essential to linear algebra and rigid body physics, also serve as examples of noncommutative operators. In my project, I study the specific type of noncommutativity where $ab = -ba$, which arises in the study of exterior algebras (involving cross products) and cohomologies in higher representation theories (with connections to knot theory).

When studying abstract algebra, we often speak of algebras as a set of generators and relations. We think of generators as building blocks for an algebra, while the relations describe how these building blocks are related. In my project, I studied the *odd nilHecke algebra*, which has generators $x_1, x_2, \ldots, x_n$ and $\partial_1, \partial_2, \ldots, \partial_{n-1}$, for a fixed positive integer n. The relations are:

1. $x_i x_j + x_j x_i = 0$ for $i \neq j$
2. $\partial_i \partial_j + \partial_j \partial_i = 0$ for $|i - j| \geq 2$
3. $\partial_i^2 = 0$
4. $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$
5. $x_i \partial_i + \partial_i x_{i+1} = 1, \partial_i x_i + x_{i+1} \partial_i = 1$
6. $x_i \partial_j + \partial_j x_i = 0$ for $i \neq j, j + 1$.

One of the wonderful things about this algebra is that its relations can be expressed with diagrams:



To decode the pictures, consider a set of n vertical strands. We define a dot on the strand i to represent x_i, and a crossing of strands i and $i+1$ to represent ∂_i. The first diagrammatic relation thus says $x_i x_j + x_j x_i = 0$, which puts the ”odd” in odd nilHecke algebra. The third relation has two crossings in the same place, so it says that $\partial_i^2 = 0$. This is the ”nil” in the nilHecke.

In my research project, I discovered novel connections between the ”odd” Dunkl operators and a recently introduced algebra called the odd nilHecke algebra. This connection results in a simple expression for the odd Dunkl Laplacian, used to study various aspects of representation theory. I also related this work to the Lie group $\mathfrak{sl}_2$, a ubiquitous structure in math and physics, by constructing modified Dunkl operators. My second major goal was to study the result of inducing a more general noncommutative multiplication $ab = qba$, for any complex number q. I introduced new nilHecke algebras in this case and developed diagrammatic rules to facilitate their study.

My first main result connects odd Dunkl operators to an algebra that seems rather different at first sight. By noticing similarities in their action on the noncommutative polynomial ring, I was able to express the previously complicated odd Dunkl operator in the much simpler, better studied, and more intuitive language of divided difference operators. Formally, we show that the odd Dunkl operator η_i can be expressed as

$$\eta_i^{\text{odd}} = t(2x_i)^{-1}(1-\tau_i) + u\sum_{k\neq i} \partial_{i,k}^{\text{odd}} s_{i,k}. \tag{2.1}$$

Here, τ_i is called an *inversion*; it sends x_i to $-x_i$ and x_j to x_j for $j \neq i$. The operator $\partial_{i,k}$ is a slightly more general version of the generators above, and $s_{i,k}$ is a *transposition* (we think of it as swapping the indices i and k). Also, t and u are nonzero constants.

I also connected odd Dunkl operators to a central figure in the theory of Lie groups, $\mathfrak{sl}_2$, thus relating them to objects and representations at the heart of abstract algebra. Discovering an action of $\mathfrak{sl}_2$ required the clever modification of the odd Dunkl operator to include a novel noncommutative derivative. We can think of $\mathfrak{sl}_2$ as a mathematical GPS; it allows us to figure out where certain operators are in relation to other ideas that are better understood. Specifically, I showed that a variant, D_i, of the odd Dunkl operator can be used to construct three operators r^2, E, and Δ that satisfy the defining relations of the Lie algebra $\mathfrak{sl}_2$:

$$r^2 = (2t)^{-1}\sum_{i=1}^{n} x_i^2 \tag{2.2}$$

$$E = \sum_{i=1}^{n} x_i p_i + \frac{n}{2} + \frac{u}{t}\sum_{k\neq i} s_{i,k} \tag{2.3}$$

$$\Delta = -(2t)^{-1}\sum_{i=1}^{n} D_i^2. \tag{2.4}$$

The general type of noncommutativity, given by $x_j x_i = q x_i x_j$, $j > i$, where q can be any complex number, turns out to be much more complicated than the "odd" study. When looking at q-symmetric functions from the perspective of algebras, I was able to generalize a diagrammatic method in order to compute an important bilinear form. This once again shows the power of diagrams in algebra!

For example, suppose we want to find $(e_1e_2e_1, e_2e_2)$, where the e_k are *elementary symmetric polynomials*. Consider a platform with one strand, a platform with two strands, and a platform with one strand, all side-by-side, to represent $e_1e_2e_1$. Below that, consider two platforms with two strands each, to represent e_2e_2. We play the game of "connecting the strands". The rules are that no strands from the same platform can intersect and that there are no triple intersections of strands. There are only four ways to connect the strands:

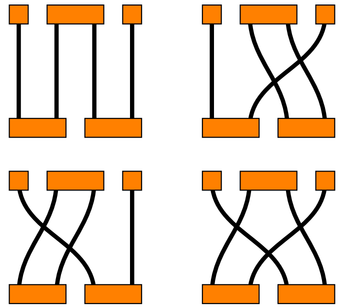


Figure 1: Diagrammatic interpretation of a bilinear form: non-commutative case.

Now count the number of times the strands intersect in each diagram. The first diagram contributes 0 intersections, the next two contribute 2 intersections each, and the last one contributes 3 intersections. Thus, $(e_1e_2e_1, e_2e_2) = q^0 + 2q^2 + q^3$. The information for a complex calculation (called a bilinear form) can be found in a really nice, visual way.

In addition, when I attacked this general case from a more algebraic standpoint, I discovered hidden versions of nilHecke algebras similar to those I had already been studying. Even though there are many ways to define such algebras, I showed that the best one leads to a new class of algebras that have very nice properties and help study q-symmetric polynomials. This is a completely new construction that exists for every value of q, and was previously only known when $q = 1$ and $q = -1$.

The motivation for this project arises from the categorification of quantum groups. A quantum group is a deformation of a group that often resembles well-known structures called semisimple Lie algebras, and they arise in representation theory and noncommutative geometry. Categorification refers to the process of obtaining a higher structure on some object; for example, replacing sets by categories, or equations by isomorphisms. In physics, categorification refers to increasing the number of dimensions, from which one obtains a new perspective on the space of lower dimensions. Similarly, categorification in mathematics increases the amount of information available about the lower structure and can play a key role in further understanding. My project relates to the new subject of odd Khovanov homology, which categorifies the Jones polynomial. Since the Jones polynomial is a well-known link invariant used in knot theory, its categorified version yields more information about the polynomial and has applications in bounding knot-theoretic numbers, studying signed hyperplane arrangements, and other topological areas.

My work on odd Dunkl operators has mathematical impact. In the commutative case, Dunkl operators are used to study important algebras in representation theory known as double affine Hecke algebas, originally introduced by Cherednik. They also relate to the study of the Calogero-Moser-Sutherland model in the study of integrable systems and the Calogero-Moser-Sutherland model in physics. As a result, my study of odd Dunkl operators contributes towards better understanding a new kind of double affine Hecke algebra, and may play a role in studying noncommutative integrable systems, as recently suggested by Bazlov and Berenstein. En route to proving a result about the odd Dunkl Laplacian, I showed that odd divided difference operators satisfy a noncommutative analog of the well-known classical Yang-Baxter equation, used in the study of statistical mechanics and knot theory. This result connects the odd nilHecke algebra to another important representation theoretic idea, thus fortifying its structure and role in abstract algebra. My result about the action of $\mathfrak{sl}_2$, which involves odd Dunkl operators also has many potential applications. The analogous result in the commutative case, for example, was used in topics ranging from Huygen's wavelet principle in physics to hypergeometric functions in math and even in studying supersymmetry. The combination of different fields involved in this project, ranging from Lie theory to low dimensional topology, results in interdisciplinary applications and an increased understanding of certain structures in abstract algebra and physics.

This project easily lends to future problems and opportunities for further research. One of the problems that I am currently studying relates to the odd Cherednik operators that I define for the first time in my paper. In the commutative case, Cherednik operators are used to study rational Cherednik algebras, useful in representation theory and in studying the quantum fractional Hall problem. Using the research on odd Dunkl operators, I am trying to find a scalar product for which the Cherednik operators are self-adjoint, since this would enable me to define odd Jack polynomials and solve a problem originally posed by my mentor. One could continue to develop the diagrammatic method in order to study relations between the elementary q-symmetric polynomials, or the new q-nilHecke algebras.

This project helps to connect math and science in a useful way. The diverse nature of representation theory ensures that a concrete theory of algebraic structures can translate into a better working knowledge of topics such as particle physics and the creation of transistors.

My research investigates new odd algebras closely related to superalgebras, which form an algebraic backbone for the study of quantum physics and supersymmetry. The odd nilHecke algebra that I study also has deep connections to the categorification of quantum groups. Taking steps towards understanding categorification can play an important role in physics, since many scientists believe that higher categories provide the proper framework for quantum gravity. As a result, the odd structures that I consider in my paper have significance in studying quantum physics and supersymmetry, which help us understand and appreciate the world around us. In addition, the q-symmetric polynomials that I consider in my paper are actually a specific case of more general structures called Calabi-Yau algebras, which are certain algebraic versions of Calabi-Yau manifolds, which serve as a model for the universe and arise in string theory!

Categorification, the motivation for my project, yields a lot of new information about the structures being categorified. The categorification of symmetric polynomials, for example, seems particularly interesting. Since symmetric polynomials can help study the fractional quantum Hall problem and lead to a greater understanding of combinatorics, topological phases, error-correction, and quantum computing, the new information that categorification provides can truly help shape the world of the future. In both representation theory and mathematical physics, the problem of studying categorifications will have a significant impact on the world in the next twenty years.

The noncommutativity that I consider in the ”odd” part of my paper also arises in parastatistics, a model of statistics closely related to the well-known Bose-Einstein model. The social benefit of better understanding these aspects of physics includes understanding the quantum fractional Hall effect (which Microsoft has invested in for technological research purposes) and in developing quantum computation, which may change the way computers function in the near future.