1 How I Got Started:

A year ago I investigated a mathematical problem relating to Latin squares. Most people, whether knowing it or not, have actually seen a Latin square at some point in their lives and many newspapers actually include partial Latin squares on a daily basis in the form of a sudoku puzzle. A Latin square is a grid of cells with numbers in each cell such that no number is repeated in any row or column, so any completed sudoku puzzle is really a 9x9 Latin square. Although Latin squares have been around for a while, providing entertainment in the form of puzzles to people ranging from Benjamin Franklin to high school students like me, there are actually quite a few open mathematical problems surrounding Latin squares. Latin squares have been used not only as puzzles, but also as tools to aid in eliminating bias in experimental design, and they are mathematically very interesting and have connections to areas like group theory and graph theory. This is why Paddy Bartlett, then a graduate student at Caltech, was interested in Latin squares and furthermore wanted to share his mathematical interest in Latin squares with a bunch of mathematically bent high school students taking part in the Canada/USA MathCamp.

Paddy taught a course on Latin squares at MathCamp 2012 and additionally offered some research opportunities to interested students. Paddy proposed a problem to me regarding the impossibility of a certain property in Latin squares of an odd size, and for the few remaining weeks of MathCamp I played around with concrete examples of Latin squares trying to get an intuition for these objects. However, like most mathematical problems, it takes a some time of just playing with the problem before one can get a feeling of what might be true and why, and even longer to construct a general proof. Therefore, as to be expected, I made little headway in the couple of weeks I had the problem at MathCamp.

Luckily, unlike a lot of lab-based research, all I really needed to continue my research was a piece of paper and a pencil. I also had the good fortune of having two math professors in my house, allowing me to bounce ideas off of my dad, and whenever I had a question, Paddy was only an email away. Continuing my research, I quickly found a paper[1] which proved what I was trying to show was impossible was actually very possible (this was probably why I made so little headway trying to prove a false claim). Clearly this paper meant that I couldn't continue my research in exactly the same way, but it didn't invalidate some of the smaller results I had already proven. For example, I showed that if you consider the first three rows of a Latin square this property I was considering actually was possible, yet the paper I found showed that this property was possible when considering the full square. This

[1]Hirschfeld, J. W. P., Magliveras, Spyros S. and Resmini, M. J. De. Interca- lates Everywhere. Geometry, Combinatorial Designs, and Related Structures: Proceedings of the First Pythagorean Conference. 245th ed. Cambridge, U.K.: Cambridge UP, 1997. 69-88. Print.

made me think that there must be a "breaking point" at some place where this property stops being impossible and becomes possible, so my research shifted from looking at Latin squares to looking at Latin rectangles, or the first few rows of a Latin square.

This paper taught me one of the most valuable lessons about mathematics: persistence is key. As poet Piet Hein says, "Problems worthy of attack prove their worth by fighting back." Finding this paper that proved the exact opposite of what I was endeavoring to prove was definitely a way that the problem fought back, but instead of letting the problem win, I persisted and used this paper, *Intercalates Everywhere*, to my advantage. As mentioned above, this new knowledge led me to a slightly different and more successful problem to investigate. Additionally, I was able to use some of the mathematical ideas in this paper to help me prove some of my own results. Even in the face of what seemed to be a roadblock, I persisted and adapted my research to ultimately yield results.

2 What I Proved:

One of the interesting and useful properties of Latin squares and Latin rectangles is the existence of "intercalates" or 2 by 2 sub squares. If we choose two rows and two columns and the four cells in the intersection of these rows and columns forms a two by two Latin square, then we have just found an intercalate! I was investigating a property called "ubiquity," meaning that every cell in the Latin square or Latin rectangle is part of an intercalate. Since intercalates are 2 by 2 sub squares, it can be expected that they would be easier to find in a Latin rectangle with an even number of columns, but the possibility of ubiquity was less clear with respect to Latin rectangles with an odd number of columns. My main question was for what values of m and n could an m by n ubiquitous Latin rectangle be constructed, or in other words, for what sizes of Latin rectangle is ubiquity possible.

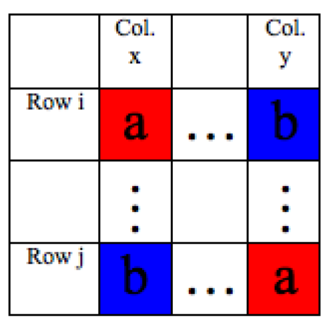

Figure 1: An intercalate in a Latin rectangle.

I proved three main results:

- It is impossible to create 3 by $2n+1$ ubiquitous Latin rectangles for all n.
- It is possible to create m by $2n$ ubiquitous Latin rectangles.
- It is possible to create $2m$ by n ubiquitous Latin rectangles for large enough values of m and n.

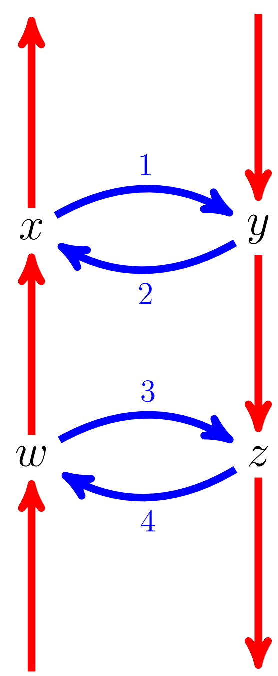

Figure 1: Two cycles of odd length (red) being "zipped" together by blue arrows.

I prove this first result by utilizing one of the connections between Latin rectangles and a special type of graph called a directed graph. Each vertex in the graph corresponds to one of the different symbols that fill the Latin rectangle. I separate the vertices into cycles of edges and use what I call a "zipper argument" to zip together and pair up all the vertices in a cycle of odd length with vertices in a separate cycle of the same odd length as shown in Figure 1. By pairing up every odd cycle using the properties of ubiquity, I show that there must be an even number of vertices, therefore showing it is impossible to have and odd number of vertices and so a 3 by odd (3 by $2n+1$) ubiquitous Latin rectangle is impossible to create.

This second result is split into two cases: skinny rectangles and fat rectangles. For the skinny rectangles, I use a technique called "quilting." I take a "patch" which is just a 2 by 2 Latin square, and I repeat this patch over and over again but with different numbers to create a quilt made up of 2 by 2 Latin squares. Clearly since the entire Latin rectangle is made up of intercalates, each cell will be in an intercalate, therefore satisfying the property of ubiquity.

1	2	3	4	5	6
2	1	4	3	6	5
3	4	5	6	1	2
4	3	6	5	2	1

Figure 3: An example of quilting.

Now for the fat rectangles, I start out with a Latin square with each cell in lots of intercalates. Then I show that if I start taking away rows, I'll be breaking up at most one intercalate for any given cell, meaning I can take away up to a certain number of rows and still have each cell in at least one intercalate. In order to construct this Latin square with each cell in lots of intercalates, I use some group theory similar to what the paper *Intercalates Everywhere* used. I establish a relationship between algebraic properties in a group and the appearance of intercalates in the group's multiplication table (Cayley table).

I continue using some group theory to prove my last result. I start out with the constructed Latin square with lots of intercalates and use a technique called "transversal expanding" and I switch around the entries in an intercalate to produce a $2m$ by $2m + 1$ ubiquitous Latin rectangle (an example is shown in Figure 4). Furthermore, I use the technique of quilting again but this time using patches of $2m$ by $2m - 1$ and $2m$ by $2m$ Latin rectangles. Finally, according to the creatively named Chicken McNugget Theorem, I can create a $2m$ by n ubiquitous Latin rectangle by quilting for large enough values of n.

R	2	3	4	5	6	7	8	9	10	11	12	1
6	1	R	3	4	5	12	7	8	9	10	11	2
5	6	1	2	R	4	11	12	7	8	9	10	3
4	5	6	1	2	12	R	11	3	7	8	9	10
3	4	5	6	1	2	9	10	R	12	7	8	11
2	3	4	5	6	1	8	9	10	11	R	7	12
7	12	11	10	9	8	1	R	5	4	3	2	6
8	7	12	11	10	9	2	1	6	R	4	3	5
9	8	7	12	11	10	3	2	1	6	5	R	4
10	R	8	7	12	11	4	3	2	1	6	5	9
11	10	9	R	7	3	5	4	12	2	1	6	8
1	9	2	8	3	7	10	6	11	5	12	4	R

Figure 4: A 12 by 13 ubiquitous Latin rectangle. Transversal expanding results in the new symbol R and the cells shown in blue show a switched intercalate.

The main goal of my research was to determine for what values of m and n I could construct an m by n ubiquitous Latin rectangle. Although I still haven't fully categorized these Latin rectangles, my results have partially resolved this question and I now have a better insight on the structure of Latin rectangles and intercalates!