

Sums involving the number of distinct prime factors function

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1 Why Math?

Math tends to scare most people - the complex formulae, the difficult notation, the hard to grasp results. But to me, math is beautiful. Starting from the simplest building blocks, you can prove something simple and beautiful and mindblowingly amazing; theorems are built on lemmas are built on more theorem are built on a minuscule number of axioms. Though my project was in number theory, every single branch of mathematics has its own fascinations.

Even when I used to think that math was beautiful because it was unambiguously, unarguably, correct - math threw a curveball at me. The first time I saw Gdels First Incompleteness Theorem in one of those books, I couldnt even wrap my head around it. It claimed that any axiomatic system dealing with the natural numbers would always contain unprovable statements. It seemed ridiculous - more than ridiculous, it seemed impossible. I scrutinized more books and articles, which is where I saw my favorite summary: This statement is false. This is breathtaking - assume that its true and the statement becomes false, assume that its false and the statement becomes true. Replace false with unprovable, analyze it formally, and voil - we have the incompleteness theorem.

This meant that there were some things that mathematicians necessarily took on faith - which seemed to defeat the entire point of math. The purpose of math was to prove things, and it fascinated me that you could prove that there would always be unanswerable questions. Above all, it captivated me that there would be a limit to the quest for knowledge.

Here - starting with four words - Gdel broke down one of the fundamental assumptions Id carefully cultivated ever since I started learning math. Math could never again prove everything; there were no longer any easy answers about truth and untruth. Yet, this is beautiful in its own way, and something which I've found repeated across each and every single field I've looked at. There are theorems so deep they seem impossible, and yet we can prove amazing things involving them. Whether it's differential forms or combinatorics or number theory, there's always something amazing underlying it.

Since my freshman year of high school, when I became increasingly bored with school math, I would look up things on my own which interested me. More than that, I would play around with them. Continued fractions? Sounds interesting - now let's try and see if I can derive a closed form if I vary the parameters this way. Sums involving the harmonic number? Let's see if I can generalize them with another parameter.

Of course, I didn't find anything truly interesting for a very long time - I would find out that what I'd done had been done two hundred years ago, in a much simpler way. But eventually, I hit gold - and it turned into this very project. I knew absolutely nothing about number theory when I started - I was working with polynomial roots at first, and only later did I realize I was staring at functions I'd seen in number theory. I decided to look into it further. The other thing I love about mathematics is that you don't need a teacher - it's built off ideas in a logical progression, and to me a well written proof comes to life and explains its own ideas to me. I started my project with wikipedia and a pirated number theory textbook, and watched as math came to life in front of me. Only later, when I had written my results up, did I approach other mathematicians about what I'd done.

And there, I found out that I'd finally found something worthwhile, something that made all the years of dead ends and frustrated rediscoveries worthwhile.

So why math? Because it's *exquisite*. Every new recurrence, every new factorization identity, is evidence that I can prove something with nothing more than my wits and a pen. Its spectacular to think about and even more breathtaking to take part in. That's the beauty of mathematics: turning simple building blocks into something towering and elegant. So go ahead - if you want to research mathematics, there's nothing stopping you but you. Read and reread every proof until you understand the ideas, and more importantly get your hands dirty. Think of ways to push what you read - add parameters, remove assumptions, make connections - and eventually, I can guarantee it'll all be worth it.

2 The Good Stuff

I want to take you firsthand through the process my research followed. My project began as an accident - I was looking at the relationships between roots of polynomials, and on a stray thought I let those roots be different prime numbers. The interesting thing about polynomial roots is that they must fulfill certain, very non-intuitive, conditions. I've given the first result that kicked everything off below.

First, like good mathematicians, we have to define everything from scratch. For the duration of this paper, the symbol $\mathbb{N} := \{1, 2, 3 \dots\}$ will denote the set of positive integers and will be referred to as the natural numbers. We also define $\mathbb{C}_\alpha := \{z \in \mathbb{C} : z \neq \alpha\}$. We let $\sum_p a_p$ and $\prod_p a_p$ denote sums and products over all primes p , beginning with $p = 2$. We let $\sum_{p|n} a_p$ and $\prod_{p|n} a_p$ denote sums and products over the distinct primes that divide a positive integer n . Finally, we let $\sum_{d|n} a_d$ and $\prod_{d|n} a_d$ denote sums and products over the positive divisors of a positive integer n , including 1 and n . For example if $n = 12$, $\prod_{d|12} a_d = a_1 a_2 a_3 a_4 a_6 a_{12}$ and $\prod_{p|12} a_p = a_2 a_3$.

Finally we establish some basic properties of $\omega(n)$, the star of this paper. It is defined to be the number of distinct prime factors of a number. For example, $\omega(5) = 1$, $\omega(10) = 2$, and $\omega(100) = \omega(2^2 5^2) = 2$. It is commonly assumed that $\omega(1) = 0$. The function $\omega(n)$ is also additive, so $\omega(nm) = \omega(n) + \omega(m)$ for any coprime n and m . In fact, other (much better) mathematicians have studied $\omega(n)$ too. One classic result is the Hardy-Ramanujan Theorem found in [1], that states $\omega(n)$ has normal order $\log \log n$. Another classic result is the Erdős-Kac Theorem found in [2], which states that $\frac{\omega(n) - \log \log n}{\sqrt{\log \log n}}$ has a standard normal distribution. With this result, Erdős and Kac singlehandedly founded the field of probabilistic number theory.

Proposition 1. *Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{C}_1$. Then*

$$\left(\prod_{i=1}^n (1 - x_i) \right) \left(\sum_{i=1}^n \frac{x_i}{x_i - 1} \right) = \sum_{k=1}^n (-1)^k k \alpha_k, \tag{2.1}$$

where

$$\alpha_k := \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k}. \tag{2.2}$$

Proof. The coefficients of each power of x can be related to its roots through Vieta's formulas. Vieta's formulas for any monic polynomial of degree n state that

$$P_n(x) = \prod_{i=1}^n (x - x_i) = \sum_{k=0}^n (-1)^{n-k} \alpha_{n-k} x^k, \tag{2.3}$$

where the $x_i \in \mathbb{C}$ are the complex roots of $P_n(x)$, $1 \leq i \leq n, i \in \mathbb{N}$, and $\alpha_0 := 1$. The α_k are called the k^{th} elementary symmetric functions of the roots. For instance,

$$P_2(x) = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1 x_2,$$

$$P_3(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3,$$

where $x_1, x_2, x_3 \in \mathbb{C}$. Taking a logarithmic derivative of the product formula of $P_n(x)$ given in (2.3), we obtain an expression for $P'_n(x)$, namely

$$\frac{d}{dx} \ln P_n(x) = \frac{d}{dx} \sum_{i=1}^n \ln(x - x_i) = \frac{P'_n(x)}{P_n(x)} = \sum_{i=1}^n \frac{1}{x - x_i}.$$

Then evaluating at $x = 1$ yields

$$P'_n(1) = \left(\prod_{i=1}^n (1 - x_i) \right) \left(\sum_{i=1}^n \frac{1}{1 - x_i} \right). \quad (2.4)$$

We can also obtain an expression for $P'_n(1)$ from taking the derivative of the sum. Thus,

$$\frac{d}{dx} P_n(x) = \frac{d}{dx} (P_n(x) - (-1)^n \alpha_n) = \sum_{k=1}^n \frac{d}{dx} ((-1)^{n-k} \alpha_{n-k} x^k) = \sum_{k=1}^n (-1)^{n-k} k \alpha_{n-k} x^{k-1}. \quad (2.5)$$

We can change the lower limit to $k = 0$, since that term is 0, and evaluate (2.5) at $x = 1$ to find that

$$P'_n(1) = \sum_{k=0}^n (-1)^{n-k} (n - (n - k)) \alpha_{n-k} = n P_n(1) - \sum_{k=0}^n (-1)^{n-k} (n - k) \alpha_{n-k}. \quad (2.6)$$

Equating (2.4) and (2.6) yields

$$\left(\prod_{i=1}^n (1 - x_i) \right) \left(\sum_{i=1}^n \frac{1}{1 - x_i} - n \right) = - \sum_{k=0}^n (-1)^{n-k} (n - k) \alpha_{n-k}.$$

Bearing in mind that $n = \sum_1^n 1$ and rewriting the sum on the right hand side gives the equation (2.1). This holds true for any collection of complex numbers x_i since an arbitrary polynomial with these roots can be constructed, as long as $x_i \neq 1$ to avoid dividing by zero. ■

Corollary 2. *Let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{C}_{-1}$. Then*

$$\left(\prod_{i=1}^n (1 + x_i) \right) \left(\sum_{i=1}^n \frac{x_i}{x_i + 1} \right) = \sum_{k=1}^n k \alpha_k. \quad (2.7)$$

Proof. We repeat the previous argument with $P_n(x) = \prod_{i=1}^n (x + x_i) = \sum_{k=0}^n \alpha_{n-k} x^k$, with the α_k defined as above. This effectively maps every x_i from (2.1) to $-x_i$. ■

This is all relatively standard, and could be done by anyone with a good knowledge of high school precalc and calculus. Now we look through the literature to try and find this result, and the funny thing is that we can't find anyone who'd actually gotten their hands dirty and done it. Now that we have a result, we have to figure out how to use it. Just writing down a formula is useless - what can it help us say? Our next idea is to let these arbitrary complex numbers be different primes, and once we start simplifying everything we realize we're looking at these funny functions we'd seen before in a number theory problem. Once we realize that, we can let these x_i be any function that multiplies together like prime numbers - these functions are called multiplicative, since $f(n)f(m) = f(nm)$ as long as n and m don't have any prime factors in common.

The following theorems follow from reinterpreting (2.1). We consider divisor sums of a multiplicative function f weighted by $\omega(n)$ and $\mu(n)$ or $|\mu(n)|$. Throughout, $f(p)$ will refer to the value of f evaluated at any prime p . We define $\sum_{p|n} g(p)$ as 0 if $n = 1$, where g is any function, not necessarily multiplicative, because 1 has no distinct prime factors. Additionally, under this convention, the theorems in this section also hold for $n = 1$.

Theorem 3. Let $f(n)$ be a multiplicative function, $n \in \mathbb{N}$, and $f(p) \neq 1$ for any prime p that divides n . Then

$$\sum_{d|n} \mu(d)\omega(d)f(d) = \left(\prod_{p|n} (1 - f(p)) \right) \left(\sum_{p|n} \frac{f(p)}{f(p) - 1} \right). \quad (2.8)$$

Proof. First take any squarefree natural number n , such that $n = \prod_{i=1}^{\omega(n)} p_i$ by the fundamental theorem of arithmetic. We then let $x_i = f(p_i)$ in (2.1), such that each x_i is an arithmetic function evaluated at each distinct prime that divides n . This corresponds to letting the roots of $P_n(x)$ be the arithmetic functions of the distinct primes that divide n . We can evaluate α_k , yielding

$$\alpha_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f(p_{i_1})f(p_{i_2}) \cdots f(p_{i_k}).$$

Since f is multiplicative and each p_i is coprime to the others by definition, we have

$$\alpha_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} f(p_{i_1}p_{i_2} \cdots p_{i_k}).$$

Now we can regard α_k as the sum of $f(n)$ evaluated at the divisors of n with k prime factors. This is because each term in α_k trivially has k prime factors, and every possible product of k primes that divide n is included in α_k . This is equivalent to partitioning the divisors of n based on their number of distinct prime factors. Then (2.1) transforms into

$$\left(\prod_{i=1}^{\omega(n)} (1 - f(p_i)) \right) \left(\sum_{i=1}^{\omega(n)} \frac{f(p_i)}{f(p_i) - 1} \right) = \sum_{d|n} \omega(d)(-1)^{\omega(d)} f(d).$$

Each divisor d is squarefree since n is squarefree, so we can replace $(-1)^{\omega(d)}$ by $\mu(d)$, where $\mu(d)$ is the Möbius function. Rewriting the product and sum over p_i , $1 \leq i \leq \omega(n)$, as a product and sum over p gives (2.8) for squarefree numbers. However, we can immediately see that if n is non-squarefree, $\mu(d)$ eliminates any non-squarefree divisors on the left hand side. Meanwhile, the right hand side is evaluated over the distinct primes that divide n so changing the multiplicities of these primes will not affect the sum in any way. Therefore (2.8) is valid for all $n \in \mathbb{N}$, which completes the proof. ■

Now this is all pretty standard, but we have to take a step back and think that this is pretty damn cool. We have this nice equation, that holds for *any* multiplicative function, and says something pretty deep about our weird little function $\omega(n)$. By specializing f we get some very pretty theorems, one of which is:

Theorem 4. Let $n \in \mathbb{N}$. Then

$$\sum_{d|n} |\mu(d)|\omega(d) = \omega(n)2^{\omega(n)-1}. \quad (2.9)$$

Proof. Substituting $f(n) = 1$ gives

$$\sum_{d|n} |\mu(d)|\omega(d) = \left(\prod_{p|n} 2 \right) \left(\sum_{p|n} \frac{1}{2} \right).$$

Since the product and sum on the right hand side are over the distinct primes that divide n , each is evaluated $\omega(n)$ times. This simplifies to

$$\sum_{d|n} |\mu(d)|\omega(d) = 2^{\omega(n)} \frac{\omega(n)}{2}.$$

■

But now, we want to see what else we can say with our polynomial factorization theorem. Now it's going to get technical because we're going to try to let n have an infinite number of prime factors, so that we can take sums and products to infinity. Again, like good mathematicians, we should give some background. For later convenience we introduce the prime zeta function, denoted by $P(s)$. We define it by

$$P(s) := \sum_p \frac{1}{p^s},$$

and note that it converges for $\Re(s) > 1$. It is an analog of the Riemann zeta function, described in [3, (25.2.1)], with the sum taken over prime numbers instead of all natural numbers. For notational convenience we also define the shifted prime zeta function $P(s, a)$ as

$$P(s, a) := \sum_p \frac{1}{p^s + a},$$

such that $P(s, 0) = P(s)$.

Lemma 5. *Let $a \in \mathbb{C}$ and $|a| < 2$. Then $P(s, a)$ converges absolutely if and only if $s \in \mathbb{C}$, $\Re(s) > 1$.*

Proof. The result follows from using a direct comparison test with $P(1)$ to prove the divergence of $P(1, a)$, then taking an absolute value to bound it above and prove absolute convergence for $\Re(s) > 1$. ■

Lemma 6. *Let $a, s, k \in \mathbb{C}$ with $|a| < 2$. Then $\sum_p \frac{p^k}{p^s + a}$ converges absolutely if and only if $\Re(s) > \max(1, 1 + \Re(k))$.*

Proof. The result follows from taking an absolute value and multiplying the top and bottom by p^{-k} , then applying Lemma 5. There are two cases based on if $\Re(k) \leq 0$ or $\Re(k) > 0$. ■

We also utilize the theory of Euler products. An Euler product, described in [3, (27.4.1-2)], is the product form of a Dirichlet series. For any multiplicative function f , it is described in [5, (11.7)] as $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} = \prod_p (1 + a_p)$, where

$$a_p := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}. \tag{2.10}$$

The left hand side is an Dirichlet series, and the right hand is an Euler product. Furthermore, letting $s = \sigma + it$, the abscissa of absolute convergence is the unique real number such that $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$ absolutely converges for $\sigma > \sigma_a$. We can ask - who cares? What the heck is this weird Dirichlet series? Why is it important? In analytic number theory, however, a Dirichlet series gives you a lot of information. If you know over what values of s it converges, you know about how fast f grows. There's a whole subfield of number theory dedicated to finding special values of L -functions, which are a special type of Dirichlet series. We're finally ready to prove our main theorem:

Theorem 7. *Let $f(n)$ be a multiplicative function, $s \in \mathbb{C}$, $a_p := \sum_{m=1}^{\infty} \frac{f(p^m)}{p^{ms}}$, and $a_p \neq -1$ for any prime p . If $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$ and $\sum_p \frac{a_p}{1+a_p}$ both converge absolutely for $\sigma > \sigma_a$, then*

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)f(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \left(\sum_p \frac{a_p}{1+a_p} \right), \tag{2.11}$$

which converges absolutely for $\sigma > \sigma_a$.

Proof. We choose a suitable x_i to substitute into (2.7), so we let $x_i = a_{p_i}$, $1 \leq i < \infty$. We still retain the condition $x_i = a_{p_i} \neq -1$ to avoid dividing by 0. Substituting this x_i into (2.7) gives

$$\sum_{k=1}^{\infty} k\alpha_k = \prod_p (1 + a_p) \sum_p \frac{a_p}{1 + a_p}.$$

The product and sum on the right now go through every prime p since each x_i is in a one-to-one correspondence with a sum over the i th prime. We also have that the product over primes is the Euler product for the Dirichlet series $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$. We now prove that α_k sums over every natural number n that has k distinct prime factors.

Lemma 8. *Let $1 \leq i < \infty$, $i \in \mathbb{N}$, p_i denote the i^{th} prime, f denote any multiplicative function, and $x_i = \sum_{m=1}^{\infty} \frac{f(p_i^m)}{p_i^{ms}}$. Furthermore, let $k \in \mathbb{N}$ and*

$$S_k := \{n \in \mathbb{N} : \omega(n) = k\}, \tag{2.12}$$

so that S_k is the set of natural numbers with k distinct prime factors. If $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$ has an abscissa of convergence σ_a , then

$$\alpha_k := \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \dots x_{i_k} = \sum_{S_k} \frac{f(n)}{n^s}, \tag{2.13}$$

which converges absolutely for $\sigma > \sigma_a$.

Proof. Let $1 \leq i \leq \infty$, $i \in \mathbb{N}$, and $1 \leq \epsilon_i < \infty$, $\epsilon_i \in \mathbb{N}$. We can directly evaluate α_k such that

$$\alpha_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \frac{f(p_{i_1}^{\epsilon_{i_1}}) f(p_{i_2}^{\epsilon_{i_2}}) \dots f(p_{i_k}^{\epsilon_{i_k}})}{\left(p_{i_1}^{\epsilon_{i_1}} p_{i_2}^{\epsilon_{i_2}} \dots p_{i_k}^{\epsilon_{i_k}}\right)^s}.$$

Here ϵ_i varies because it goes over every single power of p which is present in x_i . Since x_i is a subseries of $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$, it will also converge for $\sigma > \sigma_a$ and any manipulations involving it are valid. Since f is multiplicative and each p_i is coprime to the others by definition, we have

$$\alpha_k = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \frac{f\left(p_{i_1}^{\epsilon_{i_1}} p_{i_2}^{\epsilon_{i_2}} \dots p_{i_k}^{\epsilon_{i_k}}\right)}{\left(p_{i_1}^{\epsilon_{i_1}} p_{i_2}^{\epsilon_{i_2}} \dots p_{i_k}^{\epsilon_{i_k}}\right)^s}.$$

If we take an arbitrary natural number n with k distinct prime factors, it will be present in the sum with the k^{th} symmetric function. The k^{th} symmetric function contains every natural number with k prime factors, since k dictates the number of terms that are multiplied together to form every term in α_k . The multiplicity also doesn't matter, since that varies with ϵ_i which is independent of k .

We also know that by the fundamental theorem of arithmetic, there is a bijection between the natural numbers and the products of distinct primes with any multiplicity. This means that every product of distinct primes in the expression for α_k corresponds to a natural number n . Taking it all together shows that α_k goes over every natural number n with k distinct prime factors. Rewriting each product of primes as n then gives equation (2.13). Since $\sum_{S_k} \frac{f(n)}{n^s}$ is a subseries of $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$, it will also absolutely converge for $\sigma > \sigma_a$. ■

We then have $\sum_{k=1}^{\infty} k\alpha_k = \sum_{k=1}^{\infty} k \sum_{S_k} \frac{f(n)}{n^s}$, where S_k is defined by (2.12). This means that as k goes from 1 to ∞ the sum of each $k\alpha_k$ from the left hand side can be interpreted to go over every natural number except 1 because they have been partitioned based on how many distinct prime factors they have. The series fails to sum over $n = 1$, which does not have any prime factors, but $\omega(1) = 0$ so this does not affect the sum in any way.

We can also see $\omega(n)$ is the weight that's represented by k since we can bring it inside the inner sum as $\omega(n)$ and rewrite the double sum as a sum over the natural numbers. We can also change the bottom limit from $k = 1$ to $k = 0$ since the $k = 0$ term is 0. This gives

$$\sum_{k=1}^{\infty} k \sum_{S_k} \frac{f(n)}{n^s} = \sum_{k=0}^{\infty} \sum_{S_k} \omega(n) \frac{f(n)}{n^s} = \sum_{n \in \mathbb{N}} \frac{\omega(n) f(n)}{n^s}.$$

The inner sum converges for $\sigma > \sigma_a$, but the sum over n does not converge on this half plane in general. This rearrangement is valid if $\sum_{n \in \mathbb{N}} \frac{\omega(n) f(n)}{n^s}$ converges. However, as a Dirichlet series it is guaranteed to have an abscissa of convergence and therefore the rearrangement is valid for some σ .

Simplifying (2.7) finally shows that

$$\sum_{n \in \mathbb{N}} \frac{\omega(n) f(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \left(\sum_p \frac{a_p}{1 + a_p} \right).$$

The left hand side has the same convergence criteria as the right hand side. Therefore if $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$ has an abscissa of convergence σ_a and $\sum_p \frac{a_p}{1+a_p}$ converges absolutely for some $\sigma > \sigma_b$, the left hand side will converge for $\sigma > \max(\sigma_a, \sigma_b)$. This shows that weighting the terms of the Dirichlet series of any multiplicative function $f(n)$ by $\omega(n)$ multiplies the original series by a sum of f over primes. ■

Now, if we make our conditions on f a bit stronger, so it's more well-behaved, we can prove something even prettier:

Theorem 9. *Let $f(n)$ be a completely multiplicative function, $s \in \mathbb{C}$, and $n \in \mathbb{N}$. If $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$ has an abscissa of convergence σ_a , then*

$$\sum_{n \in \mathbb{N}} \frac{\omega(n) f(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \left(\sum_p \frac{f(p)}{p^s} \right), \tag{2.14}$$

which converges absolutely for $\sigma > \sigma_a$.

Proof. To begin, we take (2.11) but require that f be completely multiplicative. We note that since f is completely multiplicative, if $k \in \mathbb{N}$ then $f(p)^k = f(p^k)$. This means we can then sum x_i as a geometric series with common ratio $\frac{f(p_i)}{p_i^s}$. Therefore,

$$x_i = a_p = \frac{f(p_i)}{p_i^s} + \frac{f(p_i^2)}{p_i^{2s}} + \dots = \frac{f(p_i)}{p_i^s} + \frac{f(p_i)^2}{p_i^{2s}} + \dots = \frac{1}{1 - \frac{f(p_i)}{p_i^s}} - 1.$$

We also already require $\left| \frac{f(p)}{p^s} \right| < 1$, or $x_i = \frac{f(p_i)}{p_i^s} + \frac{f(p_i)^2}{p_i^{2s}} + \dots$ would diverge by the divergence test since the n^{th} term would not approach 0 as $n \rightarrow \infty$. This means that summing the geometric series is valid. We note that this also automatically satisfies the condition $x_i \neq -1$, since that would require $\left| \frac{f(p)}{p^s} \right| = 2$ which would not yield a convergent x_i . Replacing a_p by $\frac{1}{1 - \frac{f(p)}{p^s}} - 1$ in (2.11) and simplifying completes the proof. We also note that $\sum_{n \in \mathbb{N}} \frac{f(p)}{p^s}$ is a subseries of $\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}$, so it will also converge for $\sigma > \sigma_a$ and we can simplify out convergence criterion. ■

This is again a deep result, because it works for any suitably well-behaved function f . By specializing it, we can get some very pretty theorems. In the second one, we see our mysterious L -function, and have a very interesting relation for it.

Proposition 10. Let $s \in \mathbb{C}$, and $\zeta(s)$ be the Riemann zeta function. If and only if $\sigma > 1$

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)}{n^s} = \zeta(s)P(s) = \left(\sum_{n \in \mathbb{N}} \frac{1}{n^s} \right) \left(\sum_p \frac{1}{p^s} \right), \quad (2.15)$$

which converges absolutely.

Theorem 11. Let $s \in \mathbb{C}$, and let $\chi(n)$ denote a Dirichlet character. If $\sigma > 1$, then

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)\chi(n)}{n^s} = L(s, \chi) \sum_p \frac{\chi(p)}{p^s}, \quad (2.16)$$

which converges absolutely.

Theorem 12. Let $s, k \in \mathbb{C}$, and let $\sigma_k(n)$ denote the sum of the k th powers of the divisors of n . If $\sigma > \max(1, 1 + \Re(k))$, then

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)\sigma_k(n)}{n^s} = \zeta(s)\zeta(s-k) (P(s) + P(s-k) - P(2s-k)), \quad (2.17)$$

which converges absolutely.

And now, like good mathematicians, we want to show everyone else how to extend our results (even if we're too lazy to actually do it!). Therefore, we show how to generalize the methods of this paper to second and higher order derivatives. Starting with Vieta's formulas, that $P_n(x) = \prod_{i=1}^n (x - x_i) = \sum_{k=0}^n (-1)^{n-k} \alpha_{n-k} x^k$, we study the action of the differential operator $D := x \frac{d}{dx}$. Applying it once to the series gives

$$D \left(\sum_{k=0}^n (-1)^{n-k} \alpha_{n-k} x^k \right) = \sum_{k=1}^n (-1)^{n-k} k \alpha_{n-k} x^k,$$

from which we recover the familiar (2.1). Applying D a second time results in

$$D \left(\sum_{k=1}^n (-1)^{n-k} k \alpha_{n-k} x^k \right) = \sum_{k=1}^n (-1)^{n-k} k^2 \alpha_{n-k} x^k.$$

We can shift the lower index to $k = 0$ since that term will be 0, evaluate the polynomial at 1, and expand $k^2 = (n - (n - k))^2$ with the binomial expansion. This gives

$$\sum_{k=0}^n (-1)^{n-k} (n - (n - k))^2 \alpha_{n-k} = n^2 P_n(1) - 2n \sum_{k=0}^n (-1)^k k \alpha_k + \sum_{k=0}^n (-1)^k k^2 \alpha_k.$$

Applying D twice to the product form yields

$$\begin{aligned} D^2 \left(\prod_{i=1}^n (x - x_i) \right) &= D \left(x \prod_{i=1}^n (x - x_i) \sum_{i=1}^n \frac{1}{x - x_i} \right) \\ &= x \left(\prod_{i=1}^n (x - x_i) \right) \left(\sum_{i=1}^n \frac{1}{x - x_i} + x \left(\sum_{i=1}^n \frac{1}{x - x_i} \right)^2 - x \sum_{i=1}^n \left(\frac{1}{x - x_i} \right)^2 \right). \end{aligned}$$

Evaluating at $x = 1$, setting $D^2(P_n(x))$ equal for both forms of $P_n(x)$ and using (2.1) gives

$$\begin{aligned} \sum_{k=0}^n (-1)^k k^2 \alpha_k &= \left(\prod_{i=1}^n (1 - x_i) \right) \\ &\times \left(\sum_{i=1}^n \frac{1}{1 - x_i} + \left(\sum_{i=1}^n \frac{1}{1 - x_i} \right)^2 - \sum_{i=1}^n \left(\frac{1}{1 - x_i} \right)^2 - n^2 + 2n \sum_{i=1}^n \frac{x_i}{x_i - 1} \right). \end{aligned} \quad (2.18)$$

Mapping x_i to $-x_i$, letting $n = \sum_1^n 1$, and using difference of two squares factorizations let us state

$$\sum_{k=0}^n k^2 \alpha_k = \left(\prod_{i=1}^n (1 + x_i) \right) \left(\left(\sum_{i=1}^n \frac{x_i}{1 + x_i} \right)^2 + \sum_{i=1}^n \frac{x_i}{(1 + x_i)^2} \right). \quad (2.19)$$

Letting $x_i = a_p := \frac{f(p_i)}{p_i^s} + \frac{f(p_i^2)}{p_i^{2s}} + \dots = \sum_{m=1}^{\infty} \frac{f(p_i^m)}{p_i^{ms}}$, $1 \leq i < \infty$, gives

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)^2 f(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \left(\left(\sum_p \frac{a_p}{1 + a_p} \right)^2 + \sum_p \frac{a_p}{(1 + a_p)^2} \right). \quad (2.20)$$

For a completely multiplicative function, summing a_p as a geometric series reduces this to

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)^2 f(n)}{n^s} = \left(\sum_{n \in \mathbb{N}} \frac{f(n)}{n^s} \right) \left(\left(\sum_p \frac{f(p)}{p^s} \right)^2 + \sum_p \frac{f(p)}{p^s} - \sum_p \left(\frac{f(p)}{p^s} \right)^2 \right). \quad (2.21)$$

As an example, letting $f(n) = 1$ gives

$$\sum_{n \in \mathbb{N}} \frac{\omega(n)^2}{n^s} = \zeta(s) (P^2(s) + P(s) - P(2s)), \quad (2.22)$$

which converges for $\Re(s) > 1$. In general, applying D to $P_n(x)$ k times will result in a Dirichlet series of the form $\sum_{n \in \mathbb{N}} \frac{\omega(n)^k f(n)}{n^s}$.

And there - we've written a paper! Now, take a step back and try and wrap your head around what you just did. $\omega(n)$ is a tricky little function, because it'll always jump around unpredictably. For example, pick a really big prime p - say one with a trillion digits. Then $\omega(p) = 1$. But maybe $p + 1$ has a million different prime factors, and then $p + 2$ is prime again. But if we look at this wonky Dirichlet series instead, we can prove some very beautiful theorems without ever knowing the individual values of $\omega(n)$.

That's a wrap, folks.

3 Next Steps

To anyone who's had their interest in number theory piqued, I would recommend looking at Apostol's Introduction to Analytic Number Theory, or any of the other references I used that are listed below. More importantly, experiment on your own!

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