Research Reflections

1 Personal Section

Prior to my Intel research project, I had tried to get a research project with several local universities. However, all of my attempts collapsed for a variety of reasons. To be honest, I had given up on trying to find research opportunities when I received a letter of acceptance from the Research Science Institute (RSI). Sponsored by the Center for Excellence for Education (CEE), RSI accepts about 50 high school students every year, typically students who have finished their junior year. At RSI, these 50 US students plus about 25 international students conduct research with faculty at Harvard, MIT, and other Boston-area institutions with topics ranging from carbon nanotubes to dark matter to Z-DNA to coding for peer-to-peer networks to graph theory.

RSI paired me with my mentor, Mr. Huadong Pang, a mathematics graduate student at MIT. On the first day of my mentorship, Mr. Pang proposed several problems for me to consider. After trying each problem for several hours, I decided on the one that would become my Intel project. I chose this problem for its inherent beauty-though it was easy to pose, the problem had no obvious answer. This problem's complexity intrigued me and was a problem that I could work on excitedly. Furthermore, this problem could be viewed from a variety of angles (e.g. mechanical, analytical, geometric, etc.). Though not all of those perspectives proved useful, the ability to approach this problem from different angles made the problem more interesting. As a side note, I had attempted (but failed spectacularly) to prove the Isoperimetric Inequality as a freshman (not knowing that it was a famous theorem). I suspect that my previous failure motivated me toward cracking this challenge.

Surprisingly, proving my results did not require much deep mathematics. The most sophisticated technique I used in my research was Lagrange multipliers. While I did learn some basic differential geometry during the course of my research, my complex approaches often did not work. Though I began my research with line integrals and an arsenal of inequality theorems, in the end, it was an elementary approach through trigonometry that allowed me to prove my results. Even more surprising, where pages of calculus failed, a short paragraph using a mechanical argument (translating/rotating regions) succeeded [1].

As I look back on my research experience now, I am truly grateful that I had the opportunity to do math research. Having never done any research prior to this project, I discovered what an amazing field mathematical research is. An indescribable feeling of exhilaration accompanies every new discovery.

For other high school students, perhaps the most important realization I had from my research experience is to remain flexible with one's approaches to research. While there certainly are times when trying harder will yield success, there are also times when a fresh approach is far more useful. In my own small research experience, there were several occasions when I continually tried to rework a method that was fundamentally unsuited for the task; only after trying a new approach did I make progress.

2 Research Section

2.1 Background

A classic geometry problem, known as the *Isoperimetric Problem* asks: given a length of string, what shape encloses maximal area [2]? Its solution is given by the *Isoperimetric Inequality*, which states that if R is a planar region of area k and perimeter ℓ, then $4\pi k \leq \ell^2$, where equality is achieved if and only if R is a circle. Thus, among all planar regions, the circle encloses maximal area [1].

In my paper, I extended the above problem as follows: what shape maximizes area that can be enclosed by a boundary of length ℓ passing through a given set of n points? The

[1] Of course, it is also possible that I could not use more complex techniques simply because I lacked the insight to properly apply them. Moreover, it is certainly possible that a generalization of my problem would require more advanced mathematics.

generalized form of the problem is rather complex. Merely finding the minimal length of string needed to pass through the n points is the equivalent of the Traveling Salesperson Problem. In my research I was only able to find results for two points, three points, and n points which form a regular n-gon. Since the entirety of my paper is close to 20 pages, I will only outline the basics of my proof below.

2.2 Two Point Case

For the two point case, without loss of generality, we can denote the distance between the two points as 1. First, we prove the following lemma.

Lemma 1. *The maximum area enclosed by a given length of string and a given line segment is the region bound by forming the string into a circular arc.* [2]

Using Lemma 1, we can show that the string must be formed into circular arcs. Thus, two circular arcs will maximize area. For $\ell \leq 2$, no area can be enclosed. For $\ell > \pi$, a circle maximizes area. For $2 < \ell \leq \pi$, since we know that the area is enclosed by two circular arcs, we can prove that two equal arcs maximizes area.

Theorem 2. *If* $2 < \ell \leq \pi$*, then two congruent circular arcs maximizes area*

Proof (Outline). By the above lemma, the region with maximal area is bound by two circular arcs, C_1 and C_2. From here, the typical approach is to define area in terms of perimeter and maximize it with calculus. However, this approach results in solving $\sin n\gamma = \gamma$ for γ and thus is extremely difficult to do symbolically. Instead, we prove that $C_1 = C_2$ in the following six steps:

1. Define the area in terms of the angles (θ_1 and θ_2) of the circular arcs. Note that if $\theta_1 = \theta_2$, then $C_1 = C_2$.

[2] i.e. The two ends of the string are attached to the two ends of the line segment.

2. Applying Lagrange multipliers, show that, at most, one configuration satisfies the resulting system.

3. Prove that there are no solutions to the Lagrangean system if $\theta_1 > \pi$ or $\theta_2 > \pi$. Notice that since $\ell \le \pi$, we cannot have $\theta_1 > \pi$ *and* $\theta_2 > \pi$. Then, prove that for $\theta_1, \theta_2 \le \pi$, only $\theta_1 = \theta_2$ solves the Lagrangean system.

4. For each ℓ, show that $\theta_1 = \theta_2$ either maximizes or minimizes area.

5. For *all* ℓ, show that $\theta_1 = \theta_2$ *always* maximizes or *always* minimizes area.

6. For $\ell = \frac{2\pi}{3}$, show that $\theta_1 = \theta_2$ maximizes area. From this calculation, induce that $\theta_1 = \theta_2$ maximizes area for all ℓ.

2.3 Three Equidistant Points

First, we normalize all distances to be 1 and again let ℓ equal the perimeter. Note that no region can be enclosed when $\ell < 3$. Also, as in the two-point case, Lemma 1 implies that circular arcs maximize the area. Thus, let C_1,C_2, and C_3 denote the three circular arcs. Additionally, for C_i, let $L(C_i)$ denote its length, θ_i denote its central angle, and r_i denote its radius. Moreover, if $r_i = r_j$ and $\theta_i = \theta_j$, then we say C_i is congruent to C_j (i.e. $C_i \cong C_j$). If $r_i = r_j$ and $\theta_i + \theta_j = 2\pi$, then we say C_i is complementary to C_j (i.e. $C_i \cong C_j^{\complement}$). Before stating the main result for this section, we define two configurations as follows:

- Equality Configuration: $C_1 \cong C_2 \cong C_3$.
- Complementary Configuration: $\left[C_1 \cong C_2 \cong C_3^{\complement}\right] \bigcap \left[\theta_1 = \theta_2 < \theta_3\right]$

Theorem 3. *Given three equidistant points (normalized such that all distances are 1) and a string of length ℓ, the Equality Configuration maximizes area when $3 \le \ell < 4.6033388488$. When $\ell \ge 4.6033388488$, the Complementary Configuration maximizes area.*

Proof (Outline). Instead of brute force calculating the optimal configuration, we use results from the two-point case simplify analysis.

1. Applying results from the two-point case, prove that any pair of arcs must either be congruent or complementary. This result holds true whenever the points for a regular n-gon.

2. Prove that for $3 \leq \ell \leq \pi + 1$, the Equality Configuration maximizes area.

3. Prove that only the Equality Configuration or Complementary Configuration can maximize area when $\ell > \pi + 1$.

4. Prove that for $\ell > \frac{3\pi}{2}$, the Complementary Configuration maximizes area. Interestingly, though not evident from its description, for $\ell > \frac{3\pi}{2}$, there is *exactly* one way to arrange the string in Complementary Configuration. Thus, though we cannot symbolically solve for the central angle in terms of ℓ using elementary functions, we *can* solve for the central angle numerically.

5. Define the area of both configurations in terms of central angles.

6. We have shown that the Complementary Configuration maximizes area for $\ell > \frac{3\pi}{2}$ and Equality Configuration maximizes area for $3 \leq \ell \leq \pi + 1$. It seems logical that the configuration that yields maximum area changes from 'Complementary' to 'Equality' at some ℓ. We might expect $\ell = \frac{3\pi}{2}$ to be the boundary point because at $\ell = \frac{3\pi}{2}$ the Complementary Configuration yields the same figure as the Equality Configuration (i.e. three semicircles). Surprisingly, this is not true. We will prove that the Complementary Configuration maximizes area whenever it exists. Consequently, the transition from Equality Configuration to Complementary Configuration does *not* occur at $\ell = \frac{3\pi}{2}$.

   However, two difficulties complicate the final case $\pi + 1 < \ell < \frac{3\pi}{2}$. First, as before, it is extremely difficult to symbolically solve for area in terms of arc length.

Second, though we have defined area for each configuration, we lack a method to compare the two areas. Both these difficulties are overcome by defining $g(x) = \frac{x}{\sin x}$ and $f(x)$ as its inverse. Using these definitions, we can write both the area of the Equality Configuration, denoted by k_{eq}, and the area of the Complementary Configuration, denoted by k_{cp}, in terms of $f(x)$. Although $f(x)$ is not a function and cannot be solved for explicitly, analyzing properties of $f(x)$ allows us to prove that for $\pi + 1 < \ell < \frac{3\pi}{2}$, the Complementary Configuration maximizes area whenever it exists (when $\ell \geq 4.603338848$). Thus, it follows that the Equality Configuration maximizes area for $3 < \ell < 4.603338848$. We prove these results in the following manner (For this entire step, all references to ℓ are restricted to $\pi + 1 < \ell < \frac{3\pi}{2}$.):

(a) Show that

$$k_{eq} = \frac{2\ell - 6\cos f\left(\frac{\ell}{3}\right)}{8\sin f\left(\frac{\ell}{3}\right)} \qquad \text{and} \qquad k_{cp} = \frac{2\ell + 2\cos f(-\ell)}{-8\sin f(-\ell)}.$$

(b) For all but one value of ℓ, prove that $f(-\ell)$ yields two values. Graphically, though the graph is not truly parabolic, its shape resembles a right-handed parabola. Since $f(-\ell)$ is shaped in such a way, the upper branch yields a value for every ℓ in the domain.

(c) Prove that the numerator of $k_{cp} >$ numerator of k_{eq}.

(d) Prove that the denominator of $k_{cp} <$ denominator of k_{eq}. Consequently, combined with the result of (6c), we know that $k_{cp} > k_{eq}$.

(e) Show that Complementary Configuration exists only when $\ell \geq 4.603338848$.

2.4 Points That Form Regular n-gons

This section is fairly similar to the three point case. First, connect the n points to form a regular n-gon, normalize all side lengths to 1, and let ℓ denote the perimeter. When $\ell < n$, no region can be enclosed. Once again, since Lemma 1 implies that circular arcs maximize area, let $C_1, C_2, \ldots, C_n$ denote the n circular arcs. Also, for C_i, let $L(C_i)$ denote its length, θ_i denote its central angle, and r_i denote its radius. Furthermore, we define two configurations as follows:

- Equality Configuration: $C_1 \cong C_2 \cong \ldots \cong C_n$.
- Complementary Configuration: $\left[C_1 \cong C_2 \cong \ldots \cong C_n^{\complement}\right] \bigcap \left[\theta_1 = \theta_2 = \ldots = \theta_{n-1} < \theta_n\right]$

The main result for this section is as follows:

Theorem 4. *Given n points that form a regular n-gon (normalized such that all side lengths are 1) and a string of length ℓ, the Equality Configuration maximizes area when $n \leq \ell \leq \pi + n - 2$. When $\ell > \frac{n\pi}{2}$, the Complementary Configuration maximizes area.*

Proof(Outline). We prove our theorem using methods similar to the proof for the three-point case.

1. Prove that for $n \leq \ell \leq \pi + n - 2$, the Equality Configuration maximizes area.
   (a) However, in relation to the n gon, the limit $\pi + n - 2$ has little connection. Thus, for a point of reference, we prove that if e is the circumference of the circumcircle, then $e \leq \pi + n - 2$.
2. Prove that only the Equality Configuration or Complementary Configuration can maximize area when $\ell > \pi + n - 2$.
3. Prove that for $\ell > \frac{n\pi}{2}$, the Complementary Configuration maximizes area.

2.5 Remarks

In this paper, we considered shapes that maximize area of isoperimetric regions with specified boundary points. More specifically, we have proven optimal shapes for two points, three equidistant points, and a partial proof for n points that form a regular n-gon. Surprisingly, in the case of three equidistant points, the transition from the Equality Configuration to the Complementary Configuration does *not* occur at $\ell = \frac{3\pi}{2}$. This suggests that there may be unforeseen connections that may play a role in other cases.

Future work on this problem might include solving for the optimal configuration for n points which form a regular n-gon for the final case of $\pi + n - 2 < \ell < \frac{n\pi}{2}$. We can show that the string must either follow the Equality or Complementary Configuration. Moreover, our results from the case of three equidistant points suggests that the Complementary Configuration may maximize area whenever it exists.

Other future work might include solving for three non-equidistant points. By Lemma 1, the region is most likely bound by circular arcs. Moreover, the technique used in Step 1 of Theorem 3 (not shown in this paper but the technique was used to compare the three equidistant point case with the two point case) can be used to extend results from three non-equidistant points to prove results for n points with convex hulls and, possibly, help in solving n points in general. We conjecture that for convex polygonal hulls, the arcs on every side of the hull have equal radii (though the central angle may be different).

We could also extend this problem into higher dimensions. By adapting Lemma 1, we know that the region will probably be bound by spherical pieces in 3D and possibly by pieces of an n-dimensional sphere for n dimensions. Additionally, a technique similar to the one that allowed us to extend results from the two point case to three equidistant point case and to the regular convex n-gonal hull case might help in higher dimensional analogies.

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References

[1] James Jessup. "The Isoperimetric Inequality." Available at `http://artsci.shu.edu/mathcs/courses/math4912/jessupja/thesis-jessupja.pdf` (2004/07/02).

[2] Eric Weisstein. "Isoperimetric Problem." Available at `http://mathworld.wolfram.com/IsoperimetricProblem.html` (2004/08/02).