John Sillcox

Personal Section

I have been interested in mathematics since kindergarten. I always loved searching for and finding patterns in sequences of numbers. I used to read whatever information I could in order to learn more about the wonders of math.

In fifth grade I was part of my school's program for gifted math students. There the teacher introduced me to a wonderful sequence of numbers called the Fibonacci sequence, named after the 13th century mathematician Leonardo da Pisa Fibonacci. She gave me a list of the first 80 or so numbers in the sequence. My first impression was an amazement at how large these numbers seemed to get, and so quickly. She showed me how the sequence was formed: each term was the sum of the two previous terms beginning with 1, 1, 2, 3, 5, 8, …. She showed me how the sequence showed up in the turns of a pinecone and a sunflower.

Then she asked me to look in this sequence and find the only numbers that were the same as their index. Those were 1 and 5. Then she asked me to find the only numbers that were the square of their index. Those were 1 and 144. Are there any other squares in this sequence? I looked at the 80 numbers in front of me and saw no others. She then told me that these were the only perfect squares in the Fibonacci sequence. Could that really be possible?

I am sure many mathematicians remember how they learned about the Fibonacci sequence. When I learned about it, I began studying it and other sequences in search of patterns. In middle school I found a way to generate each number in the fourth diagonal of Pascal's triangle before I even knew about the binomial theorem. Around the same time I found a pattern in the first six perfect numbers and used it to successfully find the seventh: 137,438,691,328.

In eighth and ninth grade I investigated prime numbers and looked for any patterns that may be hidden in their randomness. Obviously I was unsuccessful or otherwise you would have heard of me.

In tenth grade I joined my high school's research program. My teacher said that we should start to look for ideas for an experimental research project. I knew I wanted to do math research while most of my peers would be doing science research. So I looked through many math journals for topics but all the articles were on topics far beyond my knowledge. I was a bit frustrated and considered waiting to begin research to later because the research program did not require that I submit experimental research to competitions until my senior year.

Fortunately, I had developed something of a habit of investigating various mathematical problems that I came by. In this case I was investigating something related to a past International Mathematical Olympiad (IMO) question when I noticed a pattern related to the question. My investigation led to a conjecture that I was able to prove.

Now, let me talk about how I came to do the research that led to my Intel paper, "Nonagonal Numbers in the Fibonacci Sequence and Related Diophantine Equations." Anyone who wants to do math research will, I am sure, find it invaluable to take a course in number theory. That is what I did the summer after tenth grade. Part of that class was a final project that required that each student research a topic on number theory. Rather than do literature research (studying what is known about a topic and then presenting that), I decided to do my own original research and develop my own proof.

Taking a hint from my tenth grade experience, I took a question from my textbook [5] which asked the reader to consider the numbers 1 and 36. These are the first integers that are square triangular numbers, that is they are both a square number and a triangular number (I will define these later in the article). The question asked the reader to find the next one; and the next one. Then it asked how many there were (I'll give away the answer in a few paragraphs). Can you prove it?

That question was investigated and proven later in the book. I decided to ask a similar question for my research: what are the pentagonal square numbers, those that are simultaneously pentagonal and square numbers? Well, that proved to be relatively straightforward by following the book's example. The first two are 1 and 9801. I made a conjecture about the rest of them and proved it for my final project.

At the end of the summer, I became determined to do some research for eleventh grade. I quickly discovered that the results for square triangular numbers and pentagonal square numbers as well as pentagonal triangular numbers were well known (see [3]). I decided to combine all three. I asked the question: what integers are pentagonal square triangular numbers?

Well, there is an obvious one, namely 1. Are there any others though? Well, the sets of square triangular numbers, pentagonal square numbers, and pentagonal triangular numbers are all infinite. Surely I must be able to find at least one more, if not infinitely more, I thought. That only goes to show how wrong intuition can be at times. A few months later I completed my proof that the only number that is simultaneously a pentagonal, square, and triangular number is 1. The most exciting thing about this for me was that what I proved had never before been proven. Or if it had, it had never been published.

This was the second time that I had been told in one way or another that I could search and search for integers of a certain form but would find only a select few. Do you remember the square Fibonacci numbers 1 and 144?

As I geared up for the summer after eleventh grade, I searched for a mathematician to be my mentor as I did my senior year research. With little success within a hundred miles of my house, I reconnected with my number theory teacher from the previous summer, Dr. Allison Pacelli from Williams College in Massachusetts, and she said that she would love to work with me in my research. I went up from where I live in Long Island, New York to Williams College twice over the summer to work with her. Since she was familiar with my work from the previous year, she had a few papers available from which I might get my topic for research.

The paper I chose, which became my main reference for my research, proved which numbers were heptagonal Fibonacci numbers [4]. I found other similar work on the topic through MathSciNet, a database of reviews of over half a million articles on mathematics sponsored by the American Mathematical Society [2]. The heptagonal case was completed by B. Srinivasa Rao in 2003; the square case by J. H. E. Cohn in 1964; and the triangular and pentagonal cases by Ming Luo in 1989 and 1996, respectively.

In each of these cases, each author proved that there were only a finite number of integers of each form. I decided to extend their results to a higher order of polygonal number. Originally, I looked at octagonal numbers in the Fibonacci sequence. I hit a snag and decided to apply what I had learned to the nonagonal case. I slowed again but a bit of insight overcame that obstacle. Then with some perseverance and help and encouragement from my mentor, I proved that the only nonagonal Fibonacci number is 1.

I recommended earlier that anyone interested in math research should take a course in number theory. The reason is because not only did all my work depend on what I learned in that course but also because it opened up mathematical avenues that I really had no idea were there. I had the added fortune that my teacher became my research mentor. Perhaps the most important benefit that I gained from that course was the knowledge of how to create rigorous mathematical proofs. And those high school geometry "proofs" do little to help the cause. If you don't take a course that teaches you how to prove statements rigorously, and many different courses will, not only those in number theory, I would recommend that you get your hands on some proofs and read them to understand their structure and methodology. However a course is ideal.

I have since taken three math courses outside of high school as a high school student. Understand that mathematics is not simply algebra, geometry, trigonometry, statistics, and calculus. Taking these courses, doing my research and reading others' research has really helped me understand what mathematics is all about. Mathematics is a universal language. Not universal in the sense that someone in every culture can do mathematics but in the sense that *anyone* can do mathematics. Mathematics is not about using the Law of Cosines or integrating an expression. Mathematics is about understanding and proving that the Law of Cosines is correct or that integration by parts or by some other method is valid. Even more than that, mathematics is about recognizing patterns, obvious or not, and establishing their verity. Now, not everyone will become a mathematician but everyone will do math. My experiences with math may look far different from others' but mathematics' universality ensures that we are all doing it.

Now, before I move on to discuss my Intel project, let me offer some advice based on my experiences to those of you who want to do math research. First, become good friends with patience. Second, expect to be surprised by sudden breakthroughs or insights that you may have about a project. In September of my senior year, I was still working on my project. I wanted to submit it to a competition that had an October deadline. I had become very concerned that I would not be able to submit it this competition or maybe not even Intel, due in November. Then, two weeks later I had proven the theorem and two weeks after that my paper was completed, in its initial form anyway. However, I had learned to be very patient. Even though the project wrapped up in two weeks, I had been studying it, thinking about it, contemplating it, etc. for four months. Oftentimes, a research problem will take months at a minimum and years or even decades of a mathematician's time to complete.

My other recommendation for prospective math researchers is to find a mentor. You will encounter mathematics that you do not understand but that does not mean you cannot tackle a problem. A dedicated mentor can help you overcome obstacles that you may not be able to overcome or they can simply provide direction based on their experience. Since it is not always easy to find a mentor as my experience shows, I highly recommend that you begin networking now. Who do your parents or friends' parents or teachers or uncle's college roommate's second cousin know? If you can find a mathematician, it does not hurt to ask if they would be willing to work as a mentor for your research. If they can help, then great; if they can't help, try another.

Additionally, I will recommend a few resources to help in your mathematical endeavors. First of all, I would recommend purchasing a mathematical program. I have used the student version of Stephen Wolfram's *Mathematica* but I understand that others such as *Maple* are just as good. Not only can such a program be used as a calculator but also in just about anything related to mathematics, physics, chemistry, etc. Also, I would *highly* recommend that you learn how to use LaTeX. It makes it very easy to create free professional-looking mathematical and scientific writing not only in creating expressions and tables but also in creating structure within the paper. I have used and would recommend the Art of Problem Solving's tutorial on how to obtain and use LaTeX [1].

Research Section

Once again, my project is entitled "Nonagonal Numbers in the Fibonacci Sequence and Related Diophantine Equations." I have already defined the Fibonacci sequence. Let me now define nonagonal numbers by defining a more general term, polygonal numbers. The r^{th} polygonal number of order n (or simply the r^{th} n-gonal number) is given by the formula:

$$P = \tfrac{1}{2}r[(n-2)r-(n-4)],$$

where r is a positive integer.

So from this we can define any order of polygonal number with $n = 3,4,5,...$ corresponding with triangular, square, pentagonal, … numbers, respectively. So nonagonal numbers ($n = 9$) are given by the formula:

$$N = \frac{r(7r-5)}{2},$$

with the first few for $r = 1, 2, 3, 4, \ldots$ being $N = 1, 9, 24, 46, \ldots$. I gave them earlier but let me repeat the first few Fibonacci numbers: 1, 1, 2, 3, 5, 8, …. Finally, a Diophantine equation is a polynomial equation in which only integer solutions are allowed. For example, the two equations that I solved in my project are

$$4x^2 = 5y^2(7y-5)^2 \pm 16.$$

In the course of my research, I developed an original proof of a new theorem. As I stated earlier, I proved that the only nonagonal Fibonacci number is 1. To understand how I did this, first we must complete the square in the formula for a nonagonal number. Then, since we are looking at Fibonacci numbers that are also nonagonal, we replace the N in this expression with F_n. Thus, for a Fibonacci number to be a nonagonal number, the expression $56F_n + 25$ must be a perfect square (i.e. 1, 4, 9, 16, etc.).

Why do we do this? Well, as I learned in my number theory course, there is a rather simple way to determine that an expression is never a perfect square. We employ what is known as a Jacobi symbol, named after the 19th century mathematician Carl Gustav Jakob Jacobi. I will not define a Jacobi symbol here but if you want to know what it is, check out [3] or [5] or any other text on number theory for that matter. For now I will say that essentially a Jacobi symbol tests whether or not a number or an expression is related to a perfect square. If it is then it may in fact be a perfect square. If it is not then it is *never* a perfect square.

So I used the Jacobi symbol in conjunction with the expression I derived earlier to determine exactly when that expression was related to a perfect square. For the cases that it was, I then proved that the expression could be a perfect square only for a select few values. This then proved my theorem.

Now let me describe where the two Diophantine equations come from. I used a well known identity related to Fibonacci numbers but before I present it, I will define another sequence of numbers that show up in this identity. The sequence is known as the Lucas sequence, named after the 19th century mathematician Eduoard Lucas, where the n^{th} Lucas number is denoted L_n. The sequence is defined just like the Fibonacci sequence in the sense that each term is the sum of the two previous terms but it begins differently. The first few terms are 1, 3, 4, 7, 11, ….

The identity is $L_n^2 = 5F_n^2 + 4(-1)^n$. Since my theorem considered Fibonacci numbers that are also nonagonal, if we replace F_n in this identity with the formula for a nonagonal number, we develop a Diophantine equation that is solved by the results of the theorem. Multiplying through by 4 and then replacing variables with x and y by convention yields the two equations I presented earlier. The solution set of the "+16" case of the equation is $\{(\pm 2,0),(\pm 3,1)\}$ and the solution set of the "-16" case of the equation is $\{(\pm 1,1)\}$.

This concludes my work on nonagonal numbers in the Fibonacci sequence. As I continue my research now and in the future I will look at other orders of polygonal numbers in the Fibonacci sequence, hopefully taking care of the octagonal case first. I would like to study the general case so that given any order of polygonal number I can say exactly which integers are simultaneously polygonal and Fibonacci, or at least give an upper bound so that only a finite number of integers must be checked.

I thank you for reading this far. I want you to know that all my work is simply the result of a few years of dedication, something that anyone with interest can do. I hope my experience will spur you on just as so many have done for me.

References

[1] About LaTeX. Art of Problem Solving. <http://www.artofproblemsolving.com/LaTeX/AoPS_L_About.php>.

[2] MathSciNet Homepage. American Mathematical Society. <http://www.ams.org/mathscinet>.

[3] MathWorld. Wolfram Research. <http://mathworld.wolfram.com/>.

[4] Rao, B. Srinivasa. "Heptagonal Numbers in the Fibonacci Sequence and Diophantine Equations $4x^2=5y^2(5y-3)^2\pm16$." The Fibonacci Quarterly 41.5 (Nov. 2003): 414-420.

[5] Silverman, Joseph H. A Friendly Introduction to Number Theory. 2nd ed. Upper Saddle River, NJ: Prentice Hall, 2001.