1 Personal Background

Arrghh. Again I start over. I clear my mind and refocus my thoughts. "Six and six is six, seven, eight, nine, ten, eleven, twelve," I count on my fingers. This time I will not fail, like the two past times. My steel-hard will is clenched into an iron fist of resolve. My mental faculties are determined to overcome this momentous self-imposed challenge, this task of self-discovery, the answer to the question that has been plaguing me this day-what is six times six?

Perhaps I should explain the details of the situation, as I am sure that to you figuring out six times six does not seem to be much of a challenge. It certainly does not any more to me, and has not since this night. However, as my four-year old self, lying in bed, staring at the ceiling, trying to perform an operation that I only knew how to perform from the tidbits I heard my parents and brother discussing, this was quite a gargantuan task indeed. I was supposed to be getting to sleep, but not even my bedtime could interfere with my quest for knowledge. I did not let myself go to sleep that night until I had painstakingly added six and six, and six more to that, and six more to that, and six more to that, and six more to that. Even then, I was not done. I redid the entire process several times, to make sure that I was correct.

And so I was. For as long as I can remember, I have been interested in math. I have not just been interested in how math works, but why it works. I remember in third grade asking my dad to explain to me why division by zero is impossibleand understanding it enough to explain it to my teacher, who seemed to think that 3/0=0. During my sixth grade school year, I worked with my father, completely outside of school, on learning algebra, because I was frustrated and bored with the lack of challenge I found in the math taught at school. However, my dad was unable to sufficiently explain to me the "why" part on many of the concepts I inquired about, and so I did not bother to memorize a great deal of the information tossed at me. I've noticed that about myself-I really am able to do math when I understand why it works, and so I am always questing to discover the hidden reasons behind apparent truths.

My Intel project was in many ways in tune with the mathematical theme song that has been playing throughout my life, but it was also distinct in an important way. As we will discuss later, the goal of my research was to investigate both the "how" and the "why" behind a mathematical truth. In this context, my project was a simple extension of the journey I had begun that night as a four-year-old. Yet the most important aspect of my investigation was that I was no longer treading a path worn smooth by the passage of those before me. I was no longer at a junction where I could ask my parents for the answers, nor indeed any human being, living or dead (that is, through their works, not through a medium). At long last, I had reached the stage of mathematical maturity and had embarked on my own original investigation into the nature of the universe. As I progressed with my research, the spark of interest and fascination with mathematics that had been kindled earlier in my life grew exponentially into a raging wildfire.

The project itself stemmed from my interaction with Dr. Ryan Zerr, a professor at the local university. I had begun studying one-on-one with him after taking one of his classes first semester of my sophomore year. By the end of my sophomore year, I had a fanciful notion: rather than being a passive learner of mathematics, I wanted to try to create (or discover, depending on your philosophical tendencies) new inroads in the subject. I confided this idea to Dr. Zerr, with the addendum that I would love to work on any sort of project that he could suggest–my goal was to experience true mathematical development in general rather than influence any one specific segment of math in particular. After a few weeks, Dr. Zerr came across a problem posed in an article in the Mathematical Association of America's monthly journal, and he related it to me. The paper itself had been concerned with investigating what it called *difference boxes*. At the paper's conclusion, the author posed a question about how to extend his results and ideas to *difference triangles*. It was this investigation, the details of which will be discussed later, that marked my first foray into the untamed jungles of the mathematical wilderness.

My next school year was filled with intrigue, mystery, and copious quantities of hard work. Because I took the majority of my classes at the local university, every other day I would have two consecutive free hours at my high school. I took to planting myself in the school cafeteria, armed with a notebook and pencil, where I would wrestle with ideas while students around me chatted about the typical high school gossip material. At home, I spent hours either poring over my notebook or utilizing the sophisticated computing software package, Mathematica, to help me formulate conjectures or gain further insight into the problem at hand. Every week, I met with Dr. Zerr to discuss my progress and formulate a plan for the following seven days of toil. I owe Dr. Zerr tremendous gratitude for his help in guiding my research and helping with parts of the ultimate write-up.

Throughout my investigation, I found my interest in the problem only grew. Every new question that I answered seemed to pave the way for a dozen more inquiries. It was thus that when seeming disaster struck, I used the occurrence as motivation to explore these new avenues rather than succumb to discouragement.

In the December issue of the Mathematical Assocation of America's journal, there was a letter to the editor which stated that the article Dr. Zerr had read was not doing any new mathematics. Indeed, the results in that article had been discovered and published nearly fifty years ago! Rather than being a lone voyageur in the wilderness, I suddenly found that I was treading in the footsteps of those before me. Indeed, Dr. Zerr gathered other papers on this topic, which we had learned was officially titled *Ducci sequences*, it became clear that many of my results up to this point were already common knowledge. However, it also turned out that there were several results that were completely original. I used my newfound knowledge from reading the literature, combined with my own discoveries and innovations, to expand everything that I had done before. Ultimately, I managed to alter what had seemed the destruction of the edifice of my research into an event that built it into an elegant castle of original mathematical truth.

Throughout my investigation, I found that there were two key ingredients to my ultimate success. The first was passion-the love I had for the subject I was studying. It was through my natural curiosity and desire to understand that I was able to continually come up with new ideas for my research. The second was perseverance. As with many investigations, it seemed that every conception of mine that worked was born from the failures of a hundred others. Combined, these two aspects formed an unbeatable combination. It is a mixture of these two ingredients that I would recommend to anyone interested in a similar endeavor.

2 Research

Now let's talk about my research itself. The original paper that Dr. Zerr gave me posed a question along the following lines: given a triangle whose vertices are labeled with any real numbers, label the midpoint of each side with the nonnegative difference between the labels on the vertices of that side. Now, connect all of the midpoints to form another triangle, and repeat this process indefinitely (see Fig. 1). Is there a simple way to describe the "behaviors" of these so-called *difference triangles*? That is, if we are given any triangle, is there an easy way to determine ahead of time what pattern, or lack thereof, the labels will eventually adhere to?

I investigated this problem for several months, and I found a number of interesting properties and the beginnings of an answer to this question. However, when Dr. Zerr found the other papers dealing with similar questions, they dealt with these difference objects as lists of numbers, or vectors, rather than triangles or squares or n-gons. Hence, Dr. Zerr and I switched our notation to be consistent with that of the literature. In our newly discovered language, we were studying the *Ducci map*, a function that is defined for vectors of length 3 as

$$f(v_1, v_2, v_3) = (|v_1 - v_2|, |v_2 - v_3|, |v_3 - v_1|),$$

where here we use f to denote the Ducci map. Starting from a given vector, repeatedly applying the Ducci map builds a sequence known, unsurprisingly, as a *Ducci sequence*.

Most of the work that had been done in the past focused on the generalized version of the Ducci map, where $f(v_1, v_2, , v_n) = (|v_1 - v_2|, |v_2 - v_3|, ..., |v_n - v_1|)$. That is, to build a new vector from a given one, take the nonnegative difference between adjacent entries in the given vector. These other studies had mainly focused on the special case where $v_1, v_2, ..., v_n$ were all integers[1]. My plan was to move analysis of this function into a completely different realm: that of the real numbers, where my work up to this point had been.

Regardless of the terminology, my goal remained the same. I was seeking a way of describing the behavior of a Ducci sequence without actually having to calculate large amounts of the sequence. I became intrigued by the long-range, or *asymptotic*, behavior of Ducci sequences-that is, as we apply the Ducci map again and again, what do the resulting vectors look like? Do they start to have values that spiral off and grow larger without bound? Does the sequence of vectors eventually form a repeating, or *periodic*, sequence? Maybe at some point our vector reaches the trivial periodic vector-the vector consisting of all 0s. In any case, these are mathematically interesting questions. Note that a computer could never answer them in general, as they reside outside of the realm of mere brute-force computation.

The above questions, and many more, were ones that I investigated in my journey. Some answers were obvious. For example, if we look at the definition of the Ducci map, it is clear that it always spits out a vector with nonnegative entries. If we again apply the Ducci map, the largest value in the vector cannot increase, since we subtract some nonnegative number from it. This means that the vectors in a Ducci sequence cannot spiral off to infinity.

Other answers were not so obvious, but they could be found in the literature on the topic. There were a few of these that I made use of in my own work. For example, it had been proven that all periodic vectors (that is, vectors that give rise to a sequence that cycles back to its beginning, as in Fig. 2) must have all of their nonzero entries equal. Also, the only odd length vectors that can ever have a Ducci sequence containing $(0, 0, ..., 0)$ are those of the form $(a, a, ..., a)$ for some real number a.

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[1] Note that there were several, but not many, papers allowing $v_1, v_2, ..., v_n$ to be any real numbers.

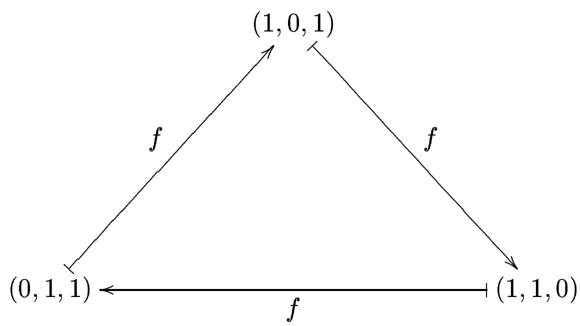

Fig. 2: Vectors in a periodic Ducci sequence.

In many of my numerical experiments, I found that the Ducci sequence under consideration often eventually reached a periodic vector, after which point it would get locked into a repeating cycle. It was not hard to show that this must happen if our starting vector has a certain simple property, which I called *homogeneity*. A vector $(v_1, v_2, ..., v_n)$ is *homogeneous* if it can be transformed into a vector with integer entries by multiplying by a nonzero number and then adding any number. For example, the vector $(\pi + \sqrt{2}, 2\pi + \sqrt{2}, 3\pi + \sqrt{2})$ is homogeneous since, upon multiplication by $\frac{1}{\pi}$ and adding $-\frac{\sqrt{2}}{\pi}$, we obtain $(1, 2, 3)$. Since for any vector $v = (v_1, v_2, ..., v_n)$, we see that

$$f(\alpha v + (\beta)_n) = f(\alpha v_1 + \beta, \alpha v_2 + \beta, ..., \alpha v_n + \beta) = |\alpha| f(v),$$

where we define $(\beta)_n = \overbrace{(\beta, \beta, ..., \beta)}^{n \text{ times}}$, it follows that any such vector behaves just like a vector with integer entries. But since an integer-entried vector only has a finite number of ways it can be put together (remember, its maximum value will not increase after the first generation), it follows that such a vector must be eventually periodic, and so must be v.

This was only half of the story. What happens with those vectors that are not homogeneous, or as I dubbed them, *heterogeneous*? This was a much harder question to answer, but it was fascinating to explore. The focus of my research became to attack the following question: which heterogeneous vectors, if any, are eventually periodic and which are not? Before we get to my methodology and results, let's discuss why this is an important question to ask in the first place.

A Ducci sequence is an example of a discrete dynamical system. That is, we can think of each vector in the sequence as a state of the system, and the Ducci map as a rule for evolving this system.

As we will show later, their behavior is extremely sensitive to initial conditions, and hence Ducci sequences can be thought of as chaotic. Such systems have been well-studied for their connections to various areas of mathematics and physics, ranging from quantum mechanics to the Riemann hypothesis. While it may not be that Ducci sequences in particular hold the key to solving problems in such areas, it is possible that the tools used to determine the structure of something like Ducci sequences could be used to gain insight into analogous systems. Thus, I set out to determine the long-range behavior of Ducci sequences, and in particular their eventual periodicity or lack thereof.

My first order of business was to develop an easy way to see if a vector is heterogeneous or not. To show a vector is homogeneous, one just needs to find the appropriate multiplier and number to add on to make all of its entries into integers. Yet showing that no such numbers exist seems a much more daunting task–there is no way to check every single pair of numbers and show that they do not suffice. Using a string of lemmas, or small results used to prove a larger result, I was able to develop a method for checking whether any vector $v = (v_1, v_2, ..., v_n)$ is heterogeneous. First, consider the vector $v' = v - (v_1)_n = (0, v_2 - v_1, ..., v_n - v_1)$. Choose any one of v''s nonzero entries and divide by it[2]. If the resulting vector has at least one irrational number, then v is heterogeneous. Otherwise, v is homogeneous. The proof that this method actually works is a good exercise for the interested reader.

Next, I came to my first of two major theorems. This first theorem, shown below, can be extended to show that any heterogeneous odd-length vector remains heterogeneous under the Ducci map.

Theorem. *Let $v = (0, v_2, ..., v_n)$ be a heterogeneous vector of odd length with nonnegative entries. Then $f(v)$ is also heterogeneous.*

Sketch of proof. It is thus sufficient to for us to show that if v is heterogeneous, then $f(v)$ is also heterogeneous. We will prove the contrapositive; that is, we show that if $f(v)$ is not heterogeneous,

[2] Note that if v' has no nonzero entries, then v must be homogeneous with all entries equal.

then v must not be heterogeneous either. To this end, suppose $f(v)$ is homogeneous. We calculate $f(v) - (v_2)_n = (0, |v_3 - v_2| - v_2, ..., |v_n| - v_2) = (0, v_2', v_3', ..., v_n') = v'$.

Case 1: Each v_i' is rational. Here is where the slick part comes into play. We define the sequence L_i, $2 \leq i < n$, as

$$L_i = \begin{cases} -1, & \text{if } v_i < v_{i+1}; \\ 1, & \text{if } v_i \geq v_{i+1}. \end{cases}$$

We see that $L_i v_i' = L_i(|v_{i+1} - v_i| - v_2) = v_i - v_{i+1} - L_i v_2$ for $2 \leq i \leq n-1$. Then consider the sum

$$L_2 v_2' + L_3 v_3' + ... + L_{n-1} v_{n-1}' + v_n'$$

This equals

$$v_2 - v_3 - L_2 v_2 + v_3 - v_4 - L_3 v_2 + ... + v_{n-1} - v_n - L_{n-1} v_1 + v_n - v_2.$$

We rearrange the above to

$$v_2 - v_3 + v_3 - v_4 + ... + v_{n-1} - v_n + v_n - v_2 - v_2(L_2 + L_3 + ... + L_{n-1}),$$

which, voila, instantly collapses to

$$-(L_2 + L_3 + ... + L_{n-1})v_2.$$

But since each $L_i v_i'$ (and v_i') is rational, it follows that their sum must also be rational. Since each L_i is either 1 or -1, and there are an odd number of them in the above sum, it follows that $L_2 + L_3 + ... + L_{n-1}$ is nonzero. Hence v_2 is rational as well. Since $v_2' = |v_3 - v_2| - v_2$ is rational, it follows that v_3 is rational as well. Similarly, $v_3' = |v_4 - v_3| - v_2$ is rational, so v_4 must be rational. In this way, we see that all of the v_i must be rational. Hence v is homogeneous, and this case is complete.

Case 2: There exists a v_i' that is irrational, where v' and each v_i' is defined as above. In this case, we can just transform v so that v' does not have any irrational entries (remember, v' is homogeneous) and go through a similar argument to Case 1.

So what does this mean? Well, since we know that an odd-length heterogeneous vector gives rise to another heterogeneous vector after applying the Ducci map, we see that no matter how many times we apply the Ducci map, a heterogeneous vector remains heterogeneous! But all periodic vectors are homogeneous. Thus, the Ducci sequence of an odd-length heterogeneous vector can never actually start repeating. However, we can use a result from the literature, which states that as we apply the Ducci map more and more times, every vector approaches a periodic one. At this point, we have a very powerful understanding of the asymptotic behavior of odd-length vectors.

But I was not finished yet. We know that heterogeneous odd-length vectors must approach a perioidic one, but which periodic one? One's intuition likely says, the zero vector! After all, aren't the entries on these vectors continually shrinking? As it turns out, this is not the case–a paper in the literature presents a vector of length 7 that asymptotically approaches, but never reaches, a nonzero periodic one. I then found examples of lengths 9 and 15. The basic idea behind how these work is the same as why the sequence $\left\{1+\dfrac{1}{2^n}\right\}$, although always shrinking, asymptotically approaches 1 rather than 0. I was not able to answer this question in general, but I was able to formulate a solution for vectors of length 3.

For a heterogeneous length three vector, $v = (v_1, v_2, v_3)$, we can greatly simplify our analysis by making a simple reduction. First, $f(v-(x)_3) = f(v)$, so we can subtract any number from all entries of v without affecting v's aymptotic behavior. A property that is unique to vectors of length 3 is that the order of their entries does not matter. Hence, we can subtract the smallest of v's entries and then order the resulting entries in increasing order. Thus, for example, we can convert $(\pi, 2, 1)$ to $(0, 1, \pi-1)$ without affecting the vector's asymptotic behavior. But such vectors will always have first entry of 0, so we only have to worry about the last two entries, namely $(1, \pi-1)$.

If we are now going to be working with ordered pairs, it makes sense we should find f's analogue on ordered pairs of numbers. That is, we want a function h that, given the ordered pair version of v, predicts the ordered pair version of $f(v)$.

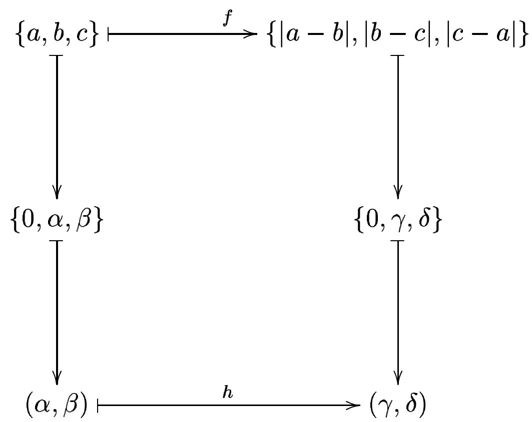

Fig. 3: The function h

It is possible to directly calculate h, and we find that

$$h(a,b) = \begin{cases} (2a-b, a), & \text{if } b < 2a; \\ (b-2a, b-a), & \text{if } b > 2a. \end{cases}$$

The rest of my paper is essentially analysis of h. I then establish three main points, which I present here without proof. In all cases suppose that (a, b) is obtained from a heterogeneous length 3 vector.

1) If $b < 2a$, then $h(a, b)$ is the same distance from the line $y = x$ as is (a, b).

2) If $b > 2a$, then $h(a, b)$ is closer to the line $y = x$ than is (a, b).

3) For any (a, b), there is an integer i such that, if we define $h^i(a, b) = (\alpha, \beta)$, $\beta > 2\alpha$.

These results are shown graphically in Fig. 4.

After some thought, one can see that the only possible asymptotic behavior consistent with rules 1), 2), 3) is if our ordered pair is spiraling down towards (0,0). Indeed, it is possible to rigorously prove that this must be the case.

This covers my work for the Intel STS. Let us recap the major points made. First, we defined a property called heterogeneity and showed how to determine if a given vector is heterogeneous or not. Secondly, we proved that all odd-length heterogeneous vectors asymptotically approach, but never actually reach, a periodic one. And thirdly, we showed that all length 3 heterogeneous vectors approach only the zero vector. Many questions remain open, such as the behavior of vectors of even length (in which case heterogeneity is no longer as powerful of a criterion for eventual periodicity; consider $(0, 1, \sqrt{2}, 1, \sqrt{2}, 1)$, which is heterogeneous but eventually periodic) and for which vector lengths all heterogeneous vectors must approach the vector of all zeros.

In total, the work was certainly demanding, but the rewards I reaped were well worth the toil. To be on the other side of the math textbook was truly an amazing experience.