Low Density Nuclear Matter in the Large N_c and Heavy Quark Limits of QCD

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1   My experience

My interest in physics was born rather serendipitously in middle school when I stumbled upon the popular physics section of my local library. In no time at all, I told you the qualitative details of Young's double slit experiment and how it contradicted certain previous classical notions of physics, but I could not have have taken a simple integral to save my life. So, due mostly to my lack of mathematical sophistication, I didn't do much physics until my junior year of high school. At that point, I realized that I *could* actually just go to the library, check out books, and learn math and physics on my own, which is exactly what I did. Although all of this self-learning was undoubtedly valuable, the one experience that was truly integral to reinforcing my interest in math and physics was the research project I conducted (and eventually submitted to both the Siemens and the Intel STS competitions) in the summer after my junior year.

I conducted the research at the University of Maryland, in the Theoretical Quarks, Hadrons, and Nuclei group (TQHN). Naturally, I was really excited to be working on real problems in theoretical physics - problems I had previously only read about. My research, which I will discuss below, focused on Quantum Chromodynamics, which is a part of the Standard Model of particle physics that deals with quarks and their interactions. Theoretical physics, in general, is deeply rooted in mathematics and as such is highly technical and not always intuitive. Luckily for me, I was already familiar with many of the qualitative aspects of the field, so the major challenge that I faced was learning the mathematics involved. I entered summer vacation with a working knowledge of multivariable calculus and linear algebra, thinking that I was going to have to learn a *lot* more math. And although I was indeed exposed to quite a few new and interesting mathematical topics, I think what was more significant was learning how to think like a physicist. This involved learning to use math in sometimes clever (and occasionally rather irresponsible) ways that conspired with physical intuition to solve mathematically non-trivial problems. This learning process is likely to be the focus of most

beginning physics researchers, as intuition and mathematical trickery grow mostly with experience.

Yet contrary to what one might think, experimenting with math in such ways doesn't lessen its beauty, but instead makes it even more elegant and impressive. To give an example, before I started working on my project linear algebra seemed to be rather technical and obscure - why should I care whether a matrix is diagonalizable or not? Why would I care what the eigenvalues of an operator are? After actively applying these concepts, however, linear algebra feels like a very intuitive and ubiquitous field.

In short, if you're a student who enjoys mathematics and wonders how research is done in the physical sciences, then don't pass up the opportunity to get some experience. Simply find a researcher who does interesting work and demonstrate to him/her your passion to learn, and most importantly, *have fun*! All else will come naturally.

Before we get into the details of my project, it will be useful to gain a rough understanding of quantum mechanics. Below, I have attempted to explore the physics in my project at a level understandable to a mathematically inclined high schooler without prior knowledge of modern physics.

2   Quantum mechanics

Quantum mechanics, developed for the most part throughout the 1920s, is a set of rules that governs the behavior of objects on atomic and subatomic scales. Unlike classical physics, in which the initial state of a physical system completely determines its subsequent time-evolution, quantum physics is inherently probabalistic. An object (*i.e.* a particle) no longer has a definite location or momentum, to name a few observables. Instead, a particle is characterized by "wavefunction", typically denoted by Ψ. This function, in conjunction with the laws of quantum mechanics can be used to find the *probability* that a particle will be found in a certain region of space, or the *probability* that a particle's momentum falls within a certain range of values. This indeterminacy is a key part of what makes quantum mechanics so unintuitive. Not all hope is lost, however, as this quantum indeterminacy evolves deterministically. Namely, given the initial wavefunction $\Psi(\vec{x}, 0)$ of a system, we can determine the wavefunction $\Psi(\vec{x}, t)$ at any point in the future using the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t}\Psi(\vec{r}, t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi(\vec{r}, t).$$

(You might see this equation with the $\hbar$ replaced with 1 - physicists like to work in *natural units* of $\hbar \equiv c \equiv 1$)

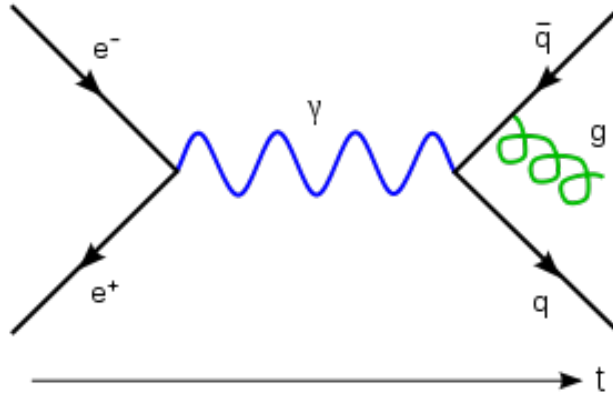Quantum mechanics is not the whole story, though - what happens to the classical notion of fields?

Figure 1: An example Feynan diagram (from Wikipedia)

3 QED and QCD

The notion of a field turns out to be incredibly useful for describing natural phenomena, as can be seen in the successes of electromagnetic theory, and thus it is natural to seek a quantum description of fields which can be used to describe interactions between particles. It turns out that any quantum field theory, or QFTs, as they're known, cannot be complete without including Einstein's theory of special relativity, which asserts the constancy of the speed of light and the coordinate invariant nature of physics - that the laws of physics should be the same in all inertial reference frames.

The virtue of QFTs is that they unite the concept of fields and particles. Quantum electrodynamics (or QED), for example, is able to explain the $\vec{E}$ and $\vec{B}$ fields, their excitations (photons), and particles such as electrons and positrons - and offers astoundingly accurate predictions. In fact, it is one of the most accurate physical theories ever created. Mathematically, QFTs are extremely complicated in general and, to the layman, are best described in terms of Feynman diagrams.

Figure 1 is a Feynman diagram representing an electron annihilating with a positron to form a photon, which, in turn, spontaneously decays into a quark and an antiquark. This representation is much more intuitive than the obscure mathematical symbols that are used for formal calculations. Additionally, Feynman diagrams are very useful for systematic approximations of QFTs known as perturbation theories. The goal of perturbation theory is to express the solution of some problem using a formal power series in some parameter λ of the theory. Mathematically, if the exact solution to a problem is X, we can write it as:

$$X = X_0 + \lambda^1 X_1 + \lambda^2 X_2 + \dots$$

Note that if the parameter λ is small, the terms with higher powers of λ (higher-order terms) are smaller compared to others. Consequently, a fairly accuarate solution can be obtained by calculating only the first few terms of the series. This technique works well when applied to QED, as the parameter is the coupling constant (what determines the strength of the force) $\alpha \approx 1/137$, which allows for remarkably accurate predictions. Unfortunately, the same technique cannot be applied to quantum chromodynamics (QCD), the theory of quarks and their interactions, as its coupling constant is *not* small.

4   Approximating QCD

So, due to the strength of the strong force that mediates quark and gluon interactions, one cannot perform a perturbation expansion on α_s. One useful approximation regime, developed by t' Hooft, considers a variant of QCD in which the number of quark colors, N_c, is taken to be large [1]. Note that this is simply a perturbation theory in $1/N_c$. This limiting case, known as large N_c QCD (although in nature, $N_c = 3$), greatly simplifies numerous aspects of the theory. Edward Witten qualitatively discusses the properties of baryons in the large N_c limit [2] and for simplicity, much of his analysis is done under the heavy quark limit, in which quark masses, M_Q, are taken to be large. He shows that the strong interaction reduces to a color-Coulomb force and that quark-antiquark pair production is suppressed as $M_Q \to \infty$. It turns out that the heavy quark limit must be imposed for calculations to be practical.

Unfortunately, the very limits that make QCD tractable render it artificial; solutions achieved in a large N_c and heavy quark world are not readily applicable to QCD. Despite this artificiality, it is generally hoped that solving QCD in an approximation scheme will provide insight into real nuclear matter.

5   Baryonic matter

The main focus of my research was to examine the properties of infinite nuclear matter in the combined large N_c and heavy quark limits. Infinite nuclear matter is an unbounded medium of baryons arranged at finite density. It is most commonly analyzed ignoring electromagnetic interactions for simplicity, and is a central problem of nuclear physics. Understanding its properties may yield qualitative and semiquantitative insights about large nuclei.

First we analyze the single baryon: it can be seen as a color-neutral (or color singlet) cluster of N_c quarks. The quarks interact with each other through a color-Coulombic $1/r$ potential. Witten argues that the large N_c limit justifies the use of a mean-field approximation. Under this approximation, each quark in a baryon
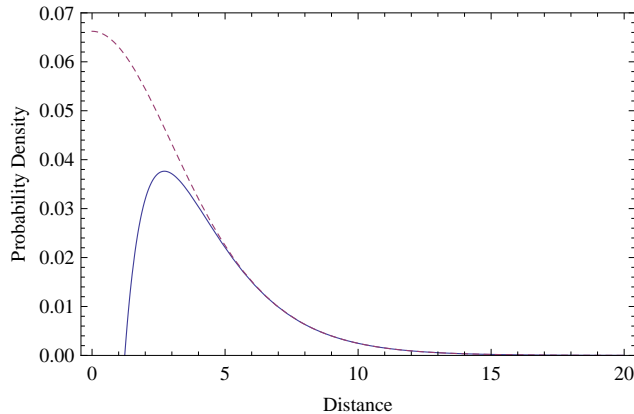
Figure 2: The dashed curve, ψ_s, is the variationally obtained approximate solution to the Schrödinger equation for the single baryon. Unfortunately it does not accurately capture the true asymptotic behavior necessary for low density computations. The solid curve, ψ_a, however, is a closed-form asymptotic solution of the Schrödinger equation that has been normalized against ψ_s.

is influenced by the average potential of all the other quarks in the baryon, and thus the potential grows with N_c. The mean-field approximation is relatively simple to calculate, and the wavefunction of a single baryon is found (via the Schrödinger equation) through a variational calculation to be

$$
\begin{aligned}
\psi_s(\vec{r}) \approx\ & 0.0876207 \exp\left(-0.0389173 r^2\right) \\
&\times \left(0.755925 - 0.00856005 r^2 + 0.000289408 r^4 \right. \\
&\quad - 3.69934 \times 10^{-6} r^6 + 3.58608 \times 10^{-8} r^8 \\
&\quad \left. -1.65987 \times 10^{-10} r^{10} + 3.64193 \times 10^{-13} r^{12}\right)
\end{aligned}
$$

This calculation was carried out a year earlier by Kamal Ndousse, who was also a Intel/Siemens semifinalist.

As this wavefunction is only approximate, it does not accurately capture the correct asymptotic behavior. The correct asymptotic behavior *can* be fairly easily determined by analytically solving the Schrödinger equation with a potential of $1/r$ and is given by

$$
\psi_a(r) = \frac{\beta}{r} \exp(-r\sqrt{-2\epsilon})\ \mathrm{U}(\frac{1}{\sqrt{-2\epsilon}},\ 0,\ 2r\sqrt{-2\epsilon})
$$

$$
\text{with } \beta \approx 0.117996
$$

where U is the confluent hypergeometric function of the second kind.

Piecing together these functions smoothly, we obtain an accurate wavefunction (Figure 2) to the single

baryon problem.

Now consider a finite density system of infinite nuclear matter in a large N_c and heavy quark world. In effect, there are N_c identical wavefunctions of quarks with different colors associated with each baryon in the system. There are two interactions to be considered in this system: the Pauli exclusion principle and the color-Coulomb strong interactions.

Since baryons are fermions, they obey the Pauli principle, and interact repulsively with each other, thus mutually orthogonalizing their wavefunctions. In this work, it will be assumed for simplicity that all quarks are in the same spin-state and that there are no flavor degeneracies.

In terms of the strong interaction, there are no direct interactions between quarks of different baryons. To see why, consider a quark, say, of color red, in an arbitrarily chosen baryon. One can imagine that from the perspective of the red quark the other quarks in the baryon collectively seem to be antired whereas other baryons appear to be color singlets. Thus the red quark will only interact with quarks in its own baryon. There are, however, attractive exchange interactions between the quarks. For example, a red quark in a baryon may emit a blue-antired gluon and turn into a antiblue quark. The gluon may then interact with an antiblue quark in a different baryon and turn it into a red quark. As each quark is now repulsed from its baryon, there is a probability of the two newly created quarks exchanging and stabilizing both baryons.

To keep the baryons from flying apart due to the Pauli principle, it is necessary to add an external force or pressure to the system; this addition induces crystallization. The situation is now analogous to that of the equation of state for nuclear matter: the matter is repulsive but the application of an external force leads to nonzero densities, similar to how the gravitational force holds together neutron stars.

The crystallization of nuclear matter is thought to occur in the large N_c limit for an intuitive physical reason: the limit's simplifications lead to static heavy baryons oscillating about potential energy minimums. Consequently, each baryon is in its minimum energy state; each state, however, has a higher energy than that of an isolated baryon due to the contributions stemming from the Pauli interactions. Evidently, the baryon energies are dependent on the crystal structure, as the lattice spacing, d, determines the strength and thus the energy of the Pauli repulsion. By construction, the baryon density ρ is low and the baryon wavefunctions are localized very large distances apart from each other. Each baryon's wavefunction drops off exponentially as seen in Eq. (5) and therefore, to leading order, the baryon is sensitive only to the presence of its nearest neighbors; even these sensitivities, however, are very small. The sensitivity is proportional to

d, which can be related to the baryon density by

$$\rho = \frac{c}{d^3} \tag{1}$$

where c is a constant that depends on the crystal structure. As the system must be in its lowest energy state, the structure must be chosen to ensure minimization of each baryon's energy. Obviously, the greater the distance between the baryons, the less they interact via the Pauli principle and so d must be maximized for any ρ for the energy to be minimized. Rearranging the above equation to

$$d = \left(\frac{c}{\rho}\right)^{1/3}$$

shows that maximizing d is equivalent to maximizing c. The problem reduces to finding a lattice configuration that maximizes c. This implies that each baryon can be treated as a rigid sphere with some maximum radius d; the maximization now becomes a close-packing problem. After some straightforward algebra, one finds that

$$c = \frac{3 V_s N}{4 \pi V_c} = \frac{3P}{4\pi}$$

where V_s is the volume of a sphere, N is the total number of spheres in a unit cell of the crystal, and V_c is the volume of the unit cell. Here the quantity $\frac{V_s N}{V_c}$ has been replaced by P, the atomic packing factor (APF), which measures the degree to which a lattice configuration is close-packed. It has been known since the time of Gauss that two configurations with the greatest APF are the hexagonal close-packed (HCP) and face-centered cubic (FCC) structures. The energy calculations performed later apply to both of these configurations as they are both energetically equivalent at leading order.

One may worry that the crystal has been implicitly assumed to be regular and that the lowest energy configuration may be irregular. On the contrary, Kepler's conjecture, put forth in 1611 and recently putatively proved by Hales, states that the greatest possible atomic packing factor is $P = \frac{\pi}{\sqrt{18}}$ *regardless* of regularity [3]. Inserting $P = \frac{\pi}{\sqrt{18}}$ into the above expression for c and inserting c into Eq. (1) yields a useful relation between the baryon density and the lattice spacing,

$$\rho = \frac{1}{4d^3\sqrt{2}}. \tag{2}$$

The quantity of interest is the interaction energy of a baryon that arises from the presence of other baryons in the lattice, as it gives us a general "feel" for how the matter behaves. To compute this quantity,

we must first find the energy of an arbitrary baryon using the following beast:

$$\langle E \rangle = - N_c \int d^3\vec{x} \, \frac{\nabla\psi^*(\vec{x})\nabla\psi(\vec{x})}{2} - \frac{N_c}{2} \iint d^3\vec{x} \, d^3\vec{y} \, \frac{\psi^*(\vec{x})\psi(\vec{x})\psi^*(\vec{y})\psi(\vec{y})}{|\vec{x}-\vec{y}|}$$
$$- \frac{N_c}{2} \sum_i \iint d^3\vec{x} \, d^3\vec{y} \, \frac{\psi^*(\vec{x})\phi_i^*(\vec{y})\psi(\vec{y})\phi_i(\vec{x}) + \psi^*(\vec{y})\phi_i^*(\vec{x})\psi(\vec{x})\phi_i(\vec{y})}{|\vec{x}-\vec{y}|}$$

where the summation is over all other baryons, ϕ_i, in the system [4]. The first term is the baryon's kinetic energy and the second and third terms are the intrabaryon and interbaryon exchange interaction energies, respectively. The behavior of the system can be found by minimizing $\langle E \rangle$ given the constraints of Pauli repulsion and normalization (*i.e.* probabilites sum to 1). This is where much of the actual derivations, etc. come in so I will simply tell you the answer.

6   Conclusion

After a few of the calculations mentioned above, we find that at leading order - first term in the N_c, M_Q and d expansions - all interactions are of negligible strength except for Pauli repulsion (primarily due to the exponentially decaying nature of the baryon wavefunctions). But clearly nuclear matter is not repulsive in reality!

Well note that the results obtained are derived from large N_c QCD at leading order, $\mathcal{O}(N_c)$. To achieve results with greater applicability to QCD, we'd have to calculate higher-order terms that correct the supposition of large N_c. Although the calculations for the $\mathcal{O}(1/N_c)$ terms are quantitatively difficult, it is believed (due to certain features of large N_c QCD's particles) that nuclear matter is self-bound once they are included.

7   Acknowledgements

References

[1] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).

[2] E. Witten, Nucl. Phys. B **160**, 57 (1979).

[3] T. C. Hales, "A Proof of the Kepler Conjecture," Math. Intell. **16**, 47 (1994).

[4] T. Cohen, N. Kumar, and K. Ndousse, arXiv:1102.2197