Patterns in the Coefficients of Powers of Polynomials Over a Finite Field

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Personal Statement

I am fascinated by problems that require a blend of computational topology, geometry, and number theory. I have also been studying fractals which interesting geometrical objects that have been used in diverse applications such as cryptography, seismology, network optimization, and even weather forecasting. However, despite the wide range of applications and interest in fractals, the general theory of these objects is still in its infancy. My work on this research project has developed some theorems and conjectures in the field of combinatorics and has begun to shed some light on some areas of fractals, one-cell automata and dynamical systems.

Historically, combinatorics has had a broad range of impact and influence on many fields in math including algebra, graph theory, probability and topology. In the last sixty years, due to the efforts of visionaries such as Rota, Stanley, and others, the field has made such significant strides in theory that it has been elevated to an independent branch of mathematics. Combinatoric optimization focuses on determining an optimal solution from among a vast set of solutions in ways that are far more efficient than just doing an exhaustive search.

This need for optimization has become increasingly more important in today's society from the perspective of both resource management as well as leveraging new opportunities. In terms of resource allocation, combinatoric optimization is being used to improve the efficiency of scheduling transportation (the traveling salesperson problem) to allocating scarce resources (such as military equipment or food distribution), through improving internet network traffic throughput, latency, and infrastructure costs. But the field has broader impact than just efficient resource allocation as it can more help in more efficiently processing large amounts of data. Increasingly, we are producing more information that we can efficiently sort through and understand, whether it is the 100k plus tweets per minute of the Presidential debates, the information gathered about global warming, or

the data mining of consumer information.

I started my research during the summer of 2012 at the Research Science Institute (RSI) at MIT. I was given the challenge of solving some open problems in the field of combinatorics that were applicable to a certain class of fractals. In order to prepare for the research, before attending RSI I self-studied some additional math such as abstract algebra and representation theory using online lectures from MIT and coaching from my RSI mentor. In addition, I had to research and understand the previous works in the field. During my time at RSI, I worked full time on the research and at the conclusion of RSI I continued the research for the following four months under the continued support of my research advisor, Professor Pavel Etingof.

My advice to students who want to undertake a project that combines science and mathematics is first and foremost to be really interested and curious about a particular problem. During the project you will likely have set backs and stumbling blocks and having that strong interest is what will help you get past any problems. Second, the way to find a project that captures your interest is to be really expansive in exploring different areas and following the paths and ideas of previous work and notice which things you wind up thinking about in your free time. Math and science are so inextricably linked that one can start almost anywhere, from a theoretical math problem to an applied problem in science, and wind up taking an interesting and exciting path through both disciplines. Along the way you can either build on previous work that interests you or formulate your own problem. Lastly, take advantage of the fact that there are so many people who are willing to help an enthusiastic researcher. Reach out to people for advice, feedback, and coaching and spend time helping others. You never know which collaboration will contribute to the success of your project.

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Abstract. We examine the behavior of the coefficients of powers of polynomials over a finite field of prime order. Extending the work of Allouche-Berthe, 1997, we study $a(n)$, the number of occurring strings of length n among coefficients of any power of a polynomial f reduced modulo a prime p. The sequence of line complexity $a(n)$ is p-regular in the sense of Allouche-Shalit. For $f = 1+x$ and general p, we derive a recursion relation for $a(n)$ then find a new formula for the generating function for $a(n)$. We use the generating function to compute the asymptotics of $a(n)/n^2$ as $n \to \infty$, which is an explicitly computable piecewise quadratic in x with $n = \lfloor p^m/x \rfloor$ and x is a real number between $1/p$ and 1. Analyzing other cases, we form a conjecture about the generating function for general $a(n)$. We examine the matrix B associated with f and p used to compute the count of a coefficient, which applies to the theory of linear cellular automata and fractals. For $p = 2$ and polynomials of small degree we compute the largest positive eigenvalue, λ, of B, related to the fractal dimension d of the corresponding fractal by $d = \log_2(\lambda)$. We find proofs and make a number of conjectures for some bounds on λ and upper bounds on its degree.

1 Introduction

It was shown by S. Wolfram and others in 1980s that 1-dimensional linear cellular automata lead at large scale to interesting examples of fractals. A basic example is the automaton associated to a polynomial f over $\mathbb{Z}/p$, whose transition matrix T_f is the matrix of multiplication by $f(x)$ on the space of Laurent polynomials in x. If $f = 1 + x$, then starting with the initial state $g_0(x) = 1$, one recovers Pascal's triangle mod p. For $p = 2$, at large scale, it produces the Sierpinski triangle shown in Figure 1. Similarly, the case of $f = 1 + x + x^2$, $p = 2$, and initial state $g_0(x) = 1$ produces the fractal shown in Figure 2.

The double sequences produced by such automata, i.e., the sequences encoding the coefficients of the powers of f, have a very interesting structure. Namely, if p is a prime, they are p-automatic sequences in the sense of [3]. In the case $f = 1+x$, this follows from Lucas' theorem that $\binom{n}{k} = \prod_i \binom{n_i}{k_i}$ mod p, where n_i, k_i are the p-ary digits of n, k.

In [6, 7], S. Wilson studied this example in the case where f is any polynomial, and computed the fractal dimension of the corresponding fractal. The answer is $\beta = \log_p(\lambda)$, where $p \leq \lambda \leq p^2$ is the largest (Perron-Frobenius) eigenvalue of a certain integer matrix B associated to f (in particular, an algebraic integer). In terms of coefficients of powers of f, this number characterizes the rate of growth of the total number of nonzero coefficients in f^i for $0 \leq i < p^n$: this number behaves like n^β. The number of nonzero coefficients of each kind can actually be computed exactly at every step of the recursion, by using a matrix method similar to Wilson's; this is explained in the paper [3].

In this paper, we compute the eigenvalues λ and their degrees for $p = 2$ for Laurent polynomials f of small degrees, observe some patterns, and make a number of conjectures (in particular, that λ can be arbitrarily close to 4) in Section 3.3. We also prove an upper bound for λ depending on the degree of f.

The size of the matrix B (which is an upper bound for the degree of λ) is the number of accessible blocks (i.e., strings that occur in the sequence of coefficients of f^i for some i) of length $deg(f)$ (for $p = 2$). This raises the question of finding the number $a(n)$ of accessible blocks of any length n. The number $a(n)$ characterizes the so-called line complexity of the corresponding linear automaton, and is studied in the paper [1]. It is shown in [1],[5], and references therein that $C_1 n^2 \leq a(n) \leq C_2 n^2$, and that for $p = 2$ and $f = 1 + x$, one has $a(n) = n^2 - n + 2$. More generally, however, the sequence $a(n)$ does not have such a simple form, even for $f = 1 + x$ and $p > 2$. The paper [1] derives a recursion for this sequence, and we derive another one in Section 2.2.1, which is equivalent. These recursions show that the sequence $a(n)$ is p-regular in the sense of [2] (the notion of p-regularity is a generalization of the notion of p-automaticity, to the case of integer, rather than mod p, values). We then proceed to find a new formula for the generating function for $a(n)$ in Section 2.3, and use it to compute the asymptotics of $a(n)/n^2$ as $n \to \infty$ in Section 2.4. It turns out that if $n = \lfloor p^m/x \rfloor$, where x is a real number between $1/p$ and 1, then $f(n)/n^2$ tends to an explicit function of x, which

is piecewise quadratic (a gluing together of 3 quadratic functions, which we explicitly compute). In Section 2.4 we also compute the maximum and minimum value of this function, which gives the best asymptotic values for C_1 and C_2. This gives us new precise results about the complexity of the Pascal triangle mod p. We also perform a similar analysis for $f = 1 + x + x^2$ and $p = 2$, and make a conjecture about the general case.

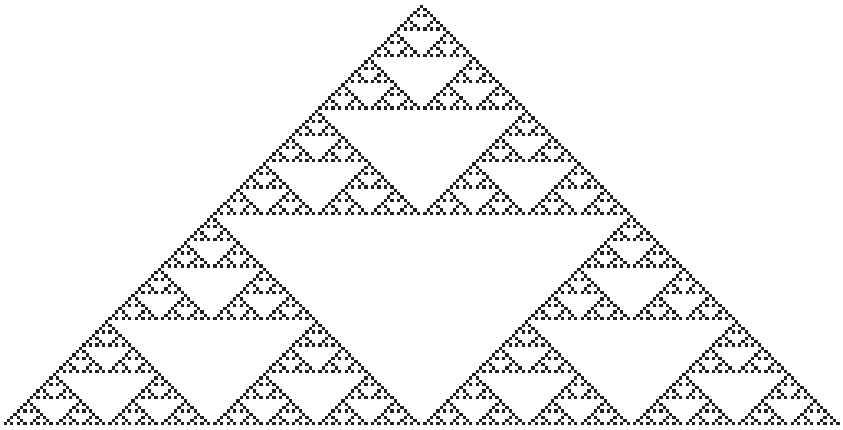


Figure 1: Fractal corresponding to $1 + x$ modulo 2 (Sierpinski's Triangle)

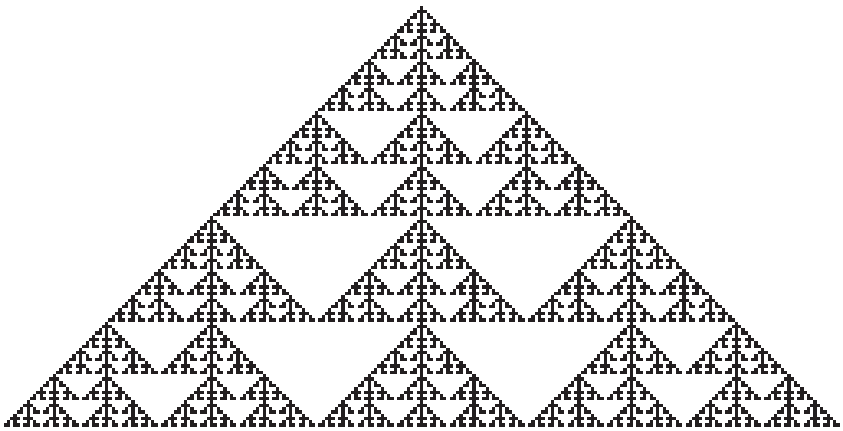


Figure 2: Fractal corresponding to $1 + x + x^2$ modulo 2

2 Accessible Blocks

2.1 Definitions

A block is a string of mod p digits. An m-block is a block with m digits. For example, the four 2-blocks modulo 2 are $00, 01, 11$, and 11.

For a polynomial $f(x)$ with integer coefficients reduced modulo p, an accessible m-block is an m-block that appears anywhere among the coefficients, ordered by powers of x, of powers of $f(x)$ modulo p. The number of accessible 0-blocks we define to be 1. Furthermore, we define row k for some $f(x)$ and p to be the coefficients of $f(x)^k$ reduced modulo p and define $a_{f(x),p}(m)$ to be the number of accessible m-blocks for the polynomial $f(x)$ and prime p.

Example 2.1. *For $f(x) = 1 + x$ and $p = 2$, the 4-blocks 1101 and 1011 are never a substring of any power of $1 + x$ reduced modulo 2. Every other 4-block appears in some power of $1 + x$ reduced modulo 2, so $a_{1+x,2}(4) = 14$.*

2.2 Recursion Relations for $a(n)$

We start with the well known fact in Lemma 2.2.

Lemma 2.2. $f(x)^{k \cdot p} \equiv f(x^p)^k \pmod{p}$.

Applying Lemma 2.2 to the accessible blocks, we have Corollary 2.3.

Corollary 2.3. *For any integer k, prime p, and polynomial $f(x)$, every row $k \cdot p$ for $f(x)$ mod p is of the form $b_1 0 \ldots 0 b_2 0 \ldots\ldots 0 b_{n-1} 0 \ldots 0 b_n$ where the entries b_i are the coefficients of $f(x)^k$, and where each string of zeros between two entries b_i and b_{i+1} is of length $p - 1$. Therefore, every accessible block from a row divisible by p is a subsection of $b_1 0 \ldots 0 b_2 0 \ldots\ldots 0 b_{n-1} 0 \ldots 0 b_n$.*

2.2.1 Accessible m-Blocks for $f(x) = 1 + x$ and General Prime p

The number of accessible m-blocks for $f(x) = 1 + x$ and any prime p, $a_{1+x,p}$, is defined by the recurrence relation in Theorem 2.4.

Theorem 2.4. *For $f(x) = 1 + x$ and any prime $p \geq 3$, for $0 \leq k \leq p-1$, the recursion relation with starting points $a_{1+x,p}(0) = 1, a_{1+x,p}(1) = p$, and $a_{1+x,p}(2) = p^2$ is*

$$\begin{aligned} a_{1+x,p}(p \cdot n + k) =& \frac{(p-k)(p-k+1)}{2} \cdot a_{1+x,p}(n) + (kp + k - k^2 + \frac{p^2 - p}{2}) \cdot a_{1+x,p}(n+1) \\ &+ \frac{k^2 - k}{2} \cdot a_{1+x,p}(n+2) - (2p-1)(2p-2). \end{aligned}$$

Proof. From Corollary 2.3, every accessible block in a row r with $r \equiv 0 \pmod{p}$ is formed by adding $p-1$ zeros between every digit of an accessible block, then adding some number of zeros (possibly none) less than p to either side. Furthermore, because $f(x) = 1+x$, the coefficient of x^i in a row is the sum modulo p of the coefficients of x^i and x^{i-1} in the previous row. Because accessible blocks are subsections of a row, any accessible m-block comes from an accessible $(m+1)$-block. Table 1 provides the general forms of the $(p \cdot n + k)$-blocks for each row modulo p. To count the multiple additions of b in the forms, we define $g_i = \binom{p-1}{i}$.

The number of accessible blocks that lead into each form in Table 1 are the triangular numbers counting downwards for $a_{1+x,p}(n)$, the triangular numbers counting upward for $a_{1+x,2}(n+2)$, and because the total number of forms is p^2, we find $a_{1+x,p}(n+1)$ through subtraction. Namely, the factor of $a_{1+x,p}(n)$ starts at p for *row* congruent to 0 modulo p and k=0, and decreases as k and *row* increase, and the coefficient of $a_{1+x,p}(n+2)$ starts at 0 for *row* congruent to 0 and 1 modulo p and increases with k and *row*. An additional $(2p-1)(2p-2)$ must be subtracted to account for blocks that satisfy multiple forms. Therefore

$$\begin{aligned} a_{1+x,p}(p \cdot n + k) =& \frac{(p-k)(p-k+1)}{2} \cdot a_{1+x,p}(n) + (kp + k - k^2 + \frac{p^2 - p}{2}) \cdot a_{1+x,p}(n+1) \\ &+ \frac{k^2 - k}{2} \cdot a_{1+x,p}(n+2) - (2p-1)(2p-2). \end{aligned}$$

□

This is equivalent to Theorem 5.10 of Allouche-Berthe [1], reproduced below in Theorem 2.5.

	Blocks for $k=$				
Row mod p	0	1	2	$\cdots$	$p-1$
0	$b_1000\ldots00b_200\ldots\ldots\ 00b_n00\ \ldots000$	b_{n+1}	0	$\cdots$	0
	$0b_100\ldots000b_20\ldots\ldots\ 000b_n0\ \ldots000$	0	b_{n+1}	$\cdots$	0
	$00b_10\ldots0000b_2\ldots\ldots\ 0000b_n\ \ldots000$	0	0	$\cdots$	0
	$\vdots\quad\vdots\quad\vdots\quad\ddots\quad\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$0000\ldots b_10000\ldots\ldots b_{n-1}0000\ldots0b_n0$	0	0	$\cdots$	b_{n+1}
	$0000\ldots0b_1000\ldots\ldots0b_{n-1}000\ldots00b_n$	0	0	$\cdots$	0
1	$b_1000\ldots0b_2b_200\ldots\ldots\ 0b_nb_n00\ \ \ldots00b_{n+1}$	b_{n+1}	0	$\cdots$	0
	$b_1b_100\ldots00b_2b_20\ldots\ldots\ 00b_nb_n0\ \ \ldots\ 000$	b_{n+1}	b_{n+1}	$\cdots$	0
	$0b_1b_10\ldots000b_2b_2\ldots\ldots\ 000b_nb_n\ \ \ldots\ 000$	0	b_{n+1}	$\cdots$	0
	$\vdots\quad\vdots\quad\vdots\quad\ddots\quad\vdots\quad\vdots\quad\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$0000\ \ldots b_10000\ldots\ldots\ b_{n-1}0000\ \ \ldots\ \ b_n0$	0	0	$\cdots$	b_{n+1}
	$0000\ \ldots b_1b_1000\ldots\ldots b_{n-1}b_{n-1}000\ldots\ b_nb_n$	0	0	$\cdots$	b_{n+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$p-1$	$b_1b_2(g_2b_2)\quad\ldots\ldots(g_4b_{n+1})(g_3b_{n+1})(g_2b_{n+1})$	b_{n+1}	b_{n+2}	$\cdots$	(g_3b_{n+2})
	$(g_2b_1)b_1b_2\quad\ldots\ldots(g_5b_{n+1})(g_4b_{n+1})(g_3b_{n+1})$	(g_2b_{n+1})	b_{n+1}	$\cdots$	(g_4b_{n+2})
	$(g_3b_1)(g_2b_1)b_1\quad\ldots\ldots(g_6b_{n+1})(g_5b_{n+1})(g_4b_{n+1})$	(g_3b_{n+1})	(g_2b_{n+1})	$\cdots$	(g_5b_{n+2})
	$\vdots\qquad\ddots\qquad\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$(g_2b_1)(g_3b_1)(g_4b_1)\ldots\ldots\quad(g_2b_n)b_nb_{n+1}$	(g_2b_{n+1})	(g_3b_{n+1})	$\cdots$	b_{n+1}
	$b_1(g_2b_1)(g_3b_1)\quad\ldots\ldots\quad(g_3b_n)(g_2b_n)b_n$	b_{n+1}	(g_2b_{n+1})	$\cdots$	(g_2b_{n+1})

Table 1: Forms of blocks for the general case $1+x$ with any prime p

Theorem 2.5. *For $0 \le k \le p-1$ and $n \ge 0$ such that $pn+k \ge 3$*

$$\begin{aligned} a_{1+x,2}(pn+k+1) - a_{1+x,2}(pn+k) =& (p-k)\Big(a_{1+x,2}(n+1) - a_{1+x,2}(n)\Big) \\ &+ k\Big(a_{1+x,2}(n+2) - a_{1+x,2}(n+1)\Big) \end{aligned}$$

with starting points $a_{1+x,2}(0)=1$, $a_{1+x,2}(1)=p$, $a_{1+x,2}(2)=p^2$, and $a_{1+x,2}(3)=\frac{p^3+4p^2-5p+2}{2}$.

2.2.2 Accessible m-Blocks for $c+x+x^2$ and prime p

Table 2 provides $a_{c+x+x^2,p}(n)$ for small n and p.

Using a method similar to the one we used for Theorem 2.4, the recursion relations appear to be those shown in Table 3.

Prime	c	a(n)
2	1	2 4 8 4 25 36 53 70 92 114
3	1	3 9 25 43 71 109 157 207 259 313
3	2	3 9 25 61 105 165 233 321 417 533
5	1	5 25 121 393 673 929 1257 1761 2341 3097
5	2	5 25 125 393 689 953 1293 1801 2389 3145
5	3	5 25 117 385 657 905 1221 1713 2277 3017
5	4	5 25 101 169 253 353 509 721 989 1313
7	1	7 49 331 1285 2137 2881 3859

Table 2: $a(n)$ for $c + x + x^2$

p	c	Recursion	k	initial
2	1	2a(n)+2a(n+1) a(n)+2a(n+1)+a(n+2)	8	1,2,4,8,14,25
3	1	6a(n)+3a(n+1) 3a(n)+6a(n+1) a(n)+7a(n+1)+a(n+2)	20	1,3,9,25
3	2	4a(n)+4a(n+1)+a(n+2) 2a(n)+5a(n+1)+2a(n+2) a(n)+4a(n+1)+4a(n+2)	32	1,3,9,25,61,105
5	1	9a(n)+12a(n+1)+4a(n+2) 6a(n)+13a(n+1)+6a(n+2) 4a(n)+12a(n+1)+9a(n+2) 2a(n)+10a(n+1)+12a(n+2)+a(n+3) a(n)+12a(n+1)+10a(n+2)+2a(n+3)	152	1,5,25,121,393,673
5	2	9a(n)+12a(n+1)+4a(n+2) 6a(n)+13a(n+1)+6a(n+2) 4a(n)+12a(n+1)+9a(n+2) 2a(n)+10a(n+1)+12a(n+2)+a(n+3) a(n)+12a(n+1)+10a(n+2)+2a(n+3)	152	1,5,25,125,393,689
5	3	9a(n)+12a(n+1)+4a(n+2) 6a(n)+13a(n+1)+6a(n+2) 4a(n)+12a(n+1)+9a(n+2) 2a(n)+10a(n+1)+12a(n+2)+a(n+3) a(n)+12a(n+1)+10a(n+2)+2a(n+3)	152	1,5,25,117,385,657
5	4	15a(n)+10a(n+1) 10a(n)+15a(n+1) 6a(n)+18a(n+1)+a(n+2) 3a(n)+19a(n+1)+3a(n+2) a(n)+18a(n+1)+6a(n+2)	72	1,5,25,101,169

Table 3: Recursions for $c + x + x^2$

We see that for $p > 2$, $a_{c+x+x^2,p}(n) = a_{1+x,p}(n)$ if $c = \frac{1}{4} \pmod p$ because $c+x+x^2 = (1+x/2)^2$. Furthermore, we arrive at Conjecture 2.6.

Conjecture 2.6. *For $c \neq \frac{1}{4} \pmod 5$, the recursion for $a_{1+x+x^2,p}(n)$ is independent of c. Only the initial terms of the recursion depend on c.*

2.3 Closed form for $a(n)$

Theorem 2.7. $a_{1+x,2}(m) = m^2 - m + 2$.

Proof. Theorem 2.4 provides the recursion relation of $a_{1+x,2}(2n) = 3a_{1+x,2}(n) + a_{1+x,2}(n+1) - 6$ and $a_{1+x,2}(n) = a_{1+x,2}(n) + 3a_{1+x,2}(n+1)$. We can find the starting points of $a_{1+x,2}(1) = 2$ and $a_{1+x,2}(2) = 4$ through inspection. This uniquely defines the sequence of accessible m-blocks. It is easy to show that the equation $a_{1+x}(m) = m^2 - m + 2$ satisfies both recursion relations through substitution, and also satisfies $a_{1+x,2}(1) = 2$ and $a_{1+x,2}(2) = 4$. □

This matches Remark 5.14 of [1].

2.3.1 Generating Functions for a(n)

Using recursion relations, we can find the generating functions $g_{f(x),p}$ for $p \geq 3$.

Theorem 2.8.

$$\begin{aligned} g_{1+x,p}(z) &= \sum_{n=0}^{\infty} a_{1+x,p}(n) z^n \\ &= \frac{1}{(1-z)^3}\Bigg(1 + (p-3)z + (p^2-3p+3)z^2 \\ &\quad + z^2 \frac{(p-1)^2}{2} \sum_{i\geq 0} \left(pz^{p^i} - 2(p-1)z^{2p^i} + (p-2)z^{3p^i}\right)\Bigg). \end{aligned}$$

Proof. We have from Theorem 2.4 that for starting points $a(0) = 1$, $a(1) = p$, and $a(2) = p^2$ the

recursion relation is defined for $pn+k>2$ as

$$\begin{aligned}a(pn+k)=&\frac{(p-k)(p-k+1)}{2}a(n)+(kp+k-k^2+\frac{p^2-p}{2})a(n+1)\\&+\frac{k^2-k}{2}a(n+2)-(2p-1)(2p-2).\end{aligned}$$

Adjusting for the $k=0,1$ cases by replacing k with $n+2$ gives

$$\begin{aligned}a(pn+k+2)=&\frac{(p-k-2)(p-k-1)}{2}a(n)+(kp-3k-k^2-2+\frac{p^2+3p}{2})a(n+1)\\&+\frac{(k+1)(k+2)}{2}a(n+2)-(2p-1)(2p-2).\end{aligned}$$

To adjust for the case when $p,k=0$, we define the recursion relation to have an additional term of $\frac{(p-2)(p-1)}{2}a(0)+\frac{(p-4)(p+1)}{2}a(1)-(2p-1)(2p-2)$ subtracted from the right hand side for only the case of $p,k=0$.

We multiply through by z^{pn+k}, then sum over $k=0$ to $p-1$, then $n=0$ to ∞. We also subtract from the right hand side of the sum the above mentioned additional term to account for the case of $p,k=0$. Defining $h(x)=\sum\limits_{n\geq 0}a(n+2)z^n$, we get

$$\begin{aligned}h(z)=&(1+z+z^2+\ldots+z^{p-1})^3h(z^p)+\frac{1}{2(1-z)^3}\Big(p^3z(1-z)^2+2p^2(1-z)(4-5z+2z^2)\\&+2(2-3z+3z^2-z^3-z^p)-p(12-19z+16z^2-5z^3-6z^p+2z^{2p})\Big)\\&-\frac{(2p-1)(2p-2)}{1-z}.\end{aligned}$$

Therefore $h(z)=\dfrac{(1-z^p)^3}{(1-z)^3}h(z^p)+Q(z)-(2p-1)(2p-2)\dfrac{1}{1-z}$ where

$$\begin{aligned}Q(z)=&\frac{1}{2(1-z)^3}\Big(p^3z(1-z)^2+2p^2(1-z)(4-5z+2z^2)+2(2-3z+3z^2-z^3-z^p)\\&-p(12-19z+16z^2-5z^3-6z^p+2z^{2p})\Big).\end{aligned}$$

We then define $u(z)=(1-z)^3h(z)$ and $R(z)=Q(z)(1-z)^3-(2p-1)(2p-2)(1-z)^2$. Iteratively

substituting gives $u(z) = u(z^{p^\infty}) + \sum_{i\geq 0} R(z^{p^i}) = a(2) + \sum_{i\geq 0} R(z^{p^i})$, or $h(z) = \frac{1}{(1-z)^3}\Big(a(2) + \sum_{i\geq 0} R(z^{p^i})\Big)$. Note that

$$\begin{aligned}\sum_{i\geq 0} R(z^{p^i}) =& \sum_{i\geq 0} \frac{1}{2}\Big((p^3 - 2p^2 - 5p + 2)z - 2(p^3 - 3p^2 + 2p - 1)z^2 \\ &+ (p-2)(p-1)^2 z^3 + 2(3p-1)z^p - 2pz^{2p}\Big) \\ =& -\Big((3p-1)z - pz^2\Big) + \frac{(p-1)^2}{2}\sum_{i\geq 0}\Big(pz^{p^i} - 2(p-1)z^{2p^i} + (p-2)z^{3p^i}\Big).\end{aligned}$$

Therefore

$$\begin{aligned}g(z) =& a(0) + a(1)z + z^2 h(z) \\ =& 1 + pz + z^2 \frac{p^2 + \sum_{i\geq 0} R(z^{p^i})}{(1-z)^3} \\ =& \frac{1 + (p-3)z + (p^2 - 3p + 3)z^2 + z^2\frac{(p-1)^2}{2}\sum_{i\geq 0}\Big(pz^{p^i} - 2(p-1)z^{2p^i} + (p-2)z^{3p^i}\Big)}{(1-z)^3}.\end{aligned}$$

□

Example 2.9. *Setting $p = 3$ in Theorem 2.8 and noting that the z^{3p^i} further reduces when $p = 3$ provides*

$$g_{1+x,3}(z) = \frac{1}{(1-z)^3}\Big(1 + 3z^2 - 2z^3 + 8z^2\sum_{i=0}^{\infty}(z^{3^i} - z^{2\cdot 3^i})\Big).$$

Example 2.10. *Setting $p = 5$ in Theorem 2.8 provides*

$$g_{1+x,5}(z) = \frac{1}{(1-z)^3}\Big(1 + 2z + 13z^2 + 8z^2\sum_{i=0}^{\infty}(5z^{5^i} - 8z^{2\cdot 5^i} + 3z^{3\cdot 5^i})\Big).$$

We can use a similar proof to find further generating functions $g_{x),p}(z)$ from the recursion relations for $a_{f(x),p}(n)$.

Theorem 2.11.

$$g_{1+x+x^2,2}(z) = \frac{1 + 2z^3 + 2z^5 - z^6 + \sum_{i=0}^{\infty}(z^{2^i} - z^{3\cdot 2^i})}{(1-z^2)(1-z)^2}.$$

Based on the recursions in Table 3 and the method provided in Theorem 2.8, we arrive at Conjecture 2.12, which is confirmed for $p = 3, 5$.

Conjecture 2.12. *For $c \neq \frac{1}{4} \pmod p$, the functional equation for the generating function $g_{c+x+x^2,p}(z)$ is*

$$g_{c+x+x^2,p}(z) = \frac{r(z^p)}{r(z)} g_{c+x+x^2,p}(z^p) - Q(z) - \frac{k}{1-z},$$

where $r(z) = (1-z^2)(1-z)^2$ and $Q(z)$ is some polynomial.

Conjecture 2.13. *For any $f(x)$ and p, the generating function $g_{f(x),p}(z)$ satisfies the equation $r(z)g_{f(x),p}(z) = r(z^p)g_{f(x),p}(z^p) + b(z)$ for some polynomials $r(z)$ and $b(z)$ depending on $f(x)$ and p.*

2.4 Limits of $\frac{a(n)}{n^2}$

Using the generating functions, we can find the asymptotic behavior of $a(n)$ as n goes to infinity. Inspired by the quadratic nature of Theorem 2.7, we examine the behavior of $\frac{a(n)}{n^2}$.

Theorem 2.14. *For $f(x) = 1 + x$ and any prime $p \geq 3$,*

$$\lim_{n\to\infty} \frac{a_{1+x,p}(n)}{n^2} = \begin{cases} \frac{p^2(p-5)(p-1)}{2(p+1)}\left(x + \frac{p+1}{p(p-5)}\right)^2 + \frac{(p-1)(p^2-7p+4)}{2(p-5)} & \frac{1}{p} \leq x \leq \frac{1}{3} \\ \frac{-(p-1)(7p^3-8p^2-9p+18)}{4(p+1)}\left(x - \frac{(p+1)(3p^2-7p+6)}{7p^3-8p^2-9p+18}\right)^2 & \\ \quad + \frac{(p-1)(p^5+5p^4-8p^3-15p^2+39p-18)}{2(7p^3-8p^2-9p+18)} & \frac{1}{3} \leq x \leq \frac{1}{2} \\ \frac{(p-2)(p-1)(p^2+2p+5)}{4(p+1)}\left(x - \frac{(p+1)^2}{p^2+2p+5}\right)^2 & \\ \quad + \frac{(p-1)(p^3+4p^2+3p-4)}{2(p^2+2p+5)} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

where $n = \lfloor \frac{p^k}{x} \rfloor$ and the limit as $n \to \infty$ is with constant x and $k \to \infty$.

Remark 2.15. *The first polynomial from Theorem 2.14 corresponding to $\frac{1}{p} \leq x \leq \frac{1}{3}$ should be understood in the sense of the limit for $p = 5$ as we divide by $(p-5)$. In this case the polynomial is not quadratic but actually the linear polynomial $20x + 8$.*

Proof. Theorem 2.8 states that

$$\begin{aligned} g(z) =& \sum_{n\geq 0} a_{1+x,p}(n) z^n \\ =& \frac{1}{(1-z)^3}\Big(1+(p-3)z+(p^2-3p+3)z^2 \\ &+ z^2\frac{(p-1)^2}{2}\sum_{i\geq 0}\Big(pz^{p^i}-2(p-1)z^{2p^i}+(p-2)z^{3p^i}\Big)\Big). \end{aligned}$$

Let $\sum_{n\geq 0} b(n)z^n = \frac{z^2}{(1-z)^3}\sum_{i\geq 0}\Big(pz^{p^i}-2(p-1)z^{2p^i}+(p-2)z^{3p^i}\Big)$.

Therefore, with the limit of $n = \lfloor \frac{p^k}{x} \rfloor \to \infty$ taken with fixed x and $k \to \infty$, we have

$$\begin{aligned} \sum_{n\geq 0} a(n)z^n =& \frac{1+(p-3)z+(p^2-3p+3)z^2}{(1-z)^3} + \sum_{n\geq 0}\frac{(p-1)^2}{2}b(n)z^n \\ \lim_{n\to\infty} a(n) =& \lim_{n\to\infty}\left(\frac{(p-1)^2n^2}{2}+\frac{(p^2-1)n}{2}+p+\frac{(p-1)^2}{2}b(n)\right) \\ \lim_{n\to\infty}\frac{a(n)}{n^2} =& \frac{(p-1)^2}{2}+\frac{(p-1)^2}{2}\lim_{n\to\infty}\frac{b(n)}{n^2}. \end{aligned}$$

Therefore, because they act similarly, we can find the asymptotics of $\frac{a(n)}{n^2}$ by understanding the behavior $\frac{b(n)}{n^2}$. We can rewrite $\sum_{n\geq 0} b(n)z^n$ as

$$\begin{aligned} \sum_{n=0}^{\infty}\Big(p\sum_{i=0}^{p^i\leq n}\frac{(n-p^i)(n-p^i-1)}{2} - 2(p-1)\sum_{i=0}^{2p^i\leq n}\frac{(n-2p^i)(n-2p^i-1)}{2} \\ +(p-2)\sum_{i=0}^{3p^i\leq n}\frac{(n-2p^i)(n-2p^i-1)}{2}z^n\Big). \end{aligned}$$

From this we see that

$$\begin{aligned} b(n) =& p\sum_{i=0}^{p^i\leq n}\left(\frac{(n-p^i)(n-p^i-1)}{2}\right) - 2(p-1)\sum_{i=0}^{2p^i\leq n}\left(\frac{(n-2p^i)(n-2p^i-1)}{2}\right) \\ &+ (p-2)\sum_{i=0}^{3p^i\leq n}\left(\frac{(n-2p^i)(n-2p^i-1)}{2}\right). \end{aligned}$$

Therefore

$$\frac{b(n)}{n^2} = \frac{p}{2}\sum_{i=0}^{p^i\le n}\left((1-\frac{p^i}{n})(1-\frac{p^i+1}{2})\right) - (p-1)\sum_{i=0}^{2p^i\le n}\left((1-\frac{2p^i}{n})(1-\frac{2p^i+1}{2})\right) + \frac{p-2}{2}\sum_{i=0}^{3p^i\le n}\left((1-\frac{3p^i}{n})(1-\frac{3p^i+1}{2})\right).$$

Let $n = \lfloor \frac{p^k}{x} \rfloor$. We can neglect the 1 in the second factor (it creates a change that goes to zero as $k \to \infty$), so we get

$$\frac{b(n)}{n^2} = \frac{p}{2}\sum_{i=0}^{p^i\le n}(1-\frac{p^i}{n})^2 - (p-1)\sum_{i=0}^{2p^i\le n}(1-\frac{2p^i}{n})^2 + \frac{p-2}{2}\sum_{i=0}^{3p^i\le n}(1-\frac{3p^i}{n})^2.$$

Note that if $x \notin [\frac{1}{3}, 1]$ then there is $m \in \mathbb{Z}$ such that $p^m x \in [\frac{1}{p}, 1]$, so we can assume $\frac{1}{p} \le x \le 1$. Ignoring the floor for simplicity, we set $n = \frac{p^k}{x}$. Therefore we get

$$\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} = \frac{p}{2}\sum_{i=0}^{p^i\le \frac{p^k}{x}}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{p^i\le \frac{p^k}{2x}}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{p^i\le \frac{p^k}{3x}}(1-3p^{i-k}x)^2.$$

When examining the upper limits of the three sums, we find that we therefore have 3 cases: $\frac{1}{p} \le x \le \frac{1}{3}, \frac{1}{3} \le x \le \frac{1}{2}, \frac{1}{2} \le x \le 1$. For the first sum, $p^i \le \frac{p^k}{x}$ gives $i \le k+1$ for $x = \frac{1}{p}$, and $i \le k$ for $x = \frac{1}{3}, \frac{1}{2}, 1$. For the second sum, $p^i \le \frac{p^k}{2x}$ gives $i \le k$ for $x = \frac{1}{p}, \frac{1}{2}, \frac{1}{3}$ and $i \le k-1$ for $x = 1$. For the third sum, $p^i \le \frac{p^k}{3x}$ gives $i \le k$ for $x = \frac{1}{p}, \frac{1}{3}$ and $i \le k-1$ for $x = \frac{1}{2}, 1$. Note that the limit is taken along the subsequences of the form $\lfloor \frac{p^k}{x} \rfloor$ with fixed x and $k \to \infty$. Also note that the limiting function does not change if x is replaced by $p \cdot x$.

For the first case of $\frac{1}{p} \leq x \leq \frac{1}{3}$, we find that

$$\begin{aligned}
\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} =& \frac{p}{2}\sum_{i=0}^{k}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{k}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{k}(1-3p^{i-k}x)^2 \\
=& (p-5)\frac{p^2-\frac{1}{p^{2k}}}{p^2-1}x^2 + 2\frac{p-\frac{1}{p^k}}{p-1}x \\
\lim_{n\to\infty}\frac{b(n)}{n^2} =& \lim_{k\to\infty}\left((p-5)\frac{p^2-\frac{1}{p^{2k}}}{p^2-1}x^2 + 2\frac{p-\frac{1}{p^k}}{p-1}x\right) \\
\lim_{n\to\infty}\frac{a(n)}{n^2} =& \frac{p^2(p-5)(p-1)}{2(p+1)}\Big(x+\frac{p+1}{p(p-5)}\Big)^2 + \frac{(p-1)(p^2-7p+4)}{2(p-5)}
\end{aligned}$$

For the case of $\frac{1}{3} \leq x \leq \frac{1}{2}$ we similarly find that because

$$\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} = \frac{p}{2}\sum_{i=0}^{k}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{k}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{k-1}(1-3p^{i-k}x)^2,$$

the limit of

$$\begin{aligned}
\lim_{n\to\infty}\frac{a(n)}{n^2} =& \frac{-(p-1)(7p^3-8p^2-9p+18)}{4(p+1)}\Big(x-\frac{(p+1)(3p^2-7p+6)}{7p^3-8p^2-9p+18}\Big)^2 - \frac{(p-4)(p-1)^2}{4} \\
&+ \frac{(p+1)(p-1)(3p^2-7p+6)^2}{4(7p^3-8p^2-9p+18)}.
\end{aligned}$$

Similarly for the case of $\frac{1}{2} \leq x \leq 1$ we find that because

$$\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} = \frac{p}{2}\sum_{i=0}^{k}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{k-1}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{k-1}(1-3p^{i-k}x)^2,$$

one has

$$\lim_{n\to\infty}\frac{a(n)}{n^2} = \frac{(p-2)(p-1)(p^2+2p+5)}{4(p+1)}\Big(x-\frac{(p+1)^2}{p^2+2p+5}\Big)^2 + \frac{(p-1)(p^3+4p^2+3p-4)}{2(p^2+2p+5)}.$$

□

Corollary 2.16. *For the polynomial* $1+x$ *and* $p \geq 3$,

$$\liminf_{n\to\infty} \frac{a_{1+x,p}(n)}{n^2} = \frac{(p-1)(p^3+4p^2+3p-4)}{2(p^2+2p+5)}$$

$$\limsup_{n\to\infty} \frac{a_{1+x,p}(n)}{n^2} = \frac{(p-1)(p^5+5p^4-8p^3-15p^2+39p-18)}{2(7p^3-8p^2-9p+18)}$$

Proof. The maximum of Theorem 2.14 is when $x = \dfrac{3p^3-4p^2-p+6}{7p^3-8p^2-9p+18}$ and the minimum is when $x = \dfrac{p^2+2p+1}{p^2+2p+5}$. □

We can also apply this to other $a_{f(x),p}(n)$.

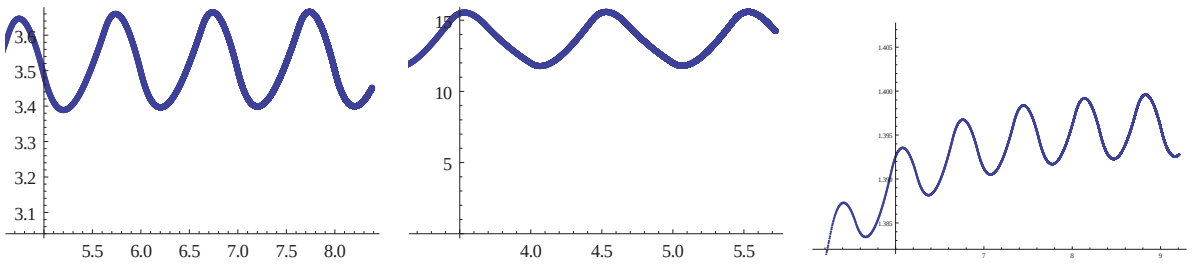
Theorem 2.17. *For polynomial* $1+x+x^2$ *and prime* 2,

$$\lim_{n\to\infty} \frac{a_{1+x+x^2,2}(n)}{n^2} = \begin{cases} \frac{5}{4} + \frac{1}{2}x - \frac{5}{12}x^2 & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{2} - \frac{1}{4}x + \frac{7}{48}x^2 & \frac{2}{3} \leq x \leq 1 \end{cases}$$

Furthermore, the upper and lower limits of $\frac{a_{1+x+x^2}(n)}{n^2}$ *are* $\dfrac{7}{5}$ *and* $\dfrac{39}{28}$ *respectively.*

The proof of Theorem 2.17 is similar to the proof of Theorem 2.14.

Using the recursion relations, we computed the upper and lower limits of $\frac{a_{f(x),p}(n)}{n^2}$ for sufficiently large n for several $f(x)$ and p, The oscillatory nature of this sequence for large n stabilizing to a periodic function in $log(x)$ is illustrated by Figure 3.



(a) $1+x \pmod 3$ (b) $1+x \pmod 5$ (c) $1+x+x^2 \pmod 2$

Figure 3: $\dfrac{a_{f(x),p}(m)}{m^2}$ with the x axis showing $\log_p m$

This matches a prior result expressed in Lemma 5.15 by [1], which states that for large n, there exists constants c_1 and c_2 such that $c_1 n^2 \leq a(n) \leq c_2 n^2$. The limits given by Corollary 2.16 provide sharp values of c_1 and c_2. [1]

3 Counting Coefficients

3.1 Definitions

For a polynomial $f(x)$, prime p, and positive integer $\alpha \leq p-1$, we define $q_{f(x),p}(k,\alpha)$ to be the number of occurrences of α among the coefficients of $f(x)^k$ reduced modulo p. Similarly, we define $q_{f(x),p}(k)$ to be the total number of nonzero coefficients of $f(x)^k$. We then define $r_{f(x),p}(n,\alpha) = \sum_{i=0}^{n-1} q_{f(x),p}(i,\alpha)$ and $r_{f(x),p}(n) = \sum_{i=0}^{n-1} q_{f(x),p}(i)$. We search for a quick method for calculating both $q_{f(x),p}(k,\alpha)$ and the asymptotic behavior of $r_{f(x),p}(n,\alpha)$ for large n.

3.2 Willson Method

Willson [6] describes an algorithm for computing the value of $r_{f(x),2}(n)$, which is provided in Theorem 3.1.

Theorem 3.1 (Willson's Method)**.** *For some polynomial $f(x)$ with maximum degree d, there exists a matrix B, row vector u, and column vector v each of size $2^d - 1$ such that $u \cdot B^k \cdot v = r_{f(x),2}(2^k)$.*

Amdeberhan-Stanley [4] describes a similar and related algorithm for calculating the number of each coefficient α for any power k for general $f(x)$ and p, namely $q_{f(x),p}(k,\alpha)$. Willson also analyzed the case of $p > 2$ in [7].

Example 3.2. *For $1+x+x^2$ mod 2, $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$. Note that the largest eigenvalue of this matrix is $1+\sqrt{5}$.*

Theorem 3.3. *The matrix B is the sum of four matrices, each of which corresponds to a self-mapping of the set $X = F_2[x]/x^d \setminus 0$.*

[1] Strictly speaking, fore these sharp values, we may not have $c_1 n^2 \leq a(n) \leq c_2 n^2$, but for any $\delta > 0$ we have $(c_1 - \delta)n^2 \leq a(n) \leq (c_2 + \delta)n^2$ for large enough n.

Theorem 3.3 follows easily from Willson [6].

Remark 3.4. *The size of the matrix B can be made smaller only by using accessible blocks, as explained in Wilson [6].*

3.3 Eigenvalue Analysis

The matrix B has nonnegative entries and is irreducible. Following Willson [6], define λ to be the Perron-Frobenius eigenvalue of B, i.e., the largest positive eigenvalue of B (it exists by the Perron-Frobenius theorem). We define $\lambda(f)$ to be the value of λ for the polynomial $f(x)$. We can approximate the value of $r_{f(x),p}(p^k, \alpha)$ with λ^k because the entries of B^k grow as a constant times λ^k.

Example 3.5. *For $f(x) = 1 + x$ and $p = 2$, $\lambda = 3$ because $B = [3]$. In this case λ corresponds exactly to the scaling of the number of nonzero coefficients when doubling the number of rows, namely $r_{1+x,2}(2k) = 3 \cdot r_{1+x,2}(k)$.*

When examining the eigenvalues, we note that there are multiple transformations of a polynomial that does not change λ.

Theorem 3.6. *We define the polynomials $f(x)$ and $g(x)$ to be similar if we can transform $f(x)$ into $g(x)$ through a combination of the transformations $f(cx)$ and $cf(x)$ with integer $1 < c < p$, $x^c f(x)$ with integer $c > 0$, $f(x^c)$ with integer $c > 1$, $x^{deg(f)} f(x^{-1})$, and $f(x)^c$ with integer $c > 1$. Any two similar polynomials have the same λ.*

Proof. Because the transformations $f(c \cdot x), f(x^c), x^c \cdot f(x), c \cdot f(x)$, and flipping a polynomial do not change the number of nonzero coefficients of a polynomial, λ do not change. Furthermore, because $f(x)^c$ is every c^{th} row, the ratios over the long term of the sums of total number of nonzero coefficients does not change, so λ is the same. Namely, let $q_{f(x)}(n)$ be the number of nonzero coefficients of $f(x)^n$. Therefore $q_{f(x)}(n+1) \leq C \cdot q_{f(x)}(n)$, where C is the number of nonzero coefficients of $f(x)$. This means that

$$r_{f(x)}(k \cdot n) = \sum_{j=0}^{k \cdot n - 1} q_{f(x)}(j) \leq \sum_{j=0}^{n-1} (1 + C + \ldots + C^{k-1}) q_{f(x)}(j \cdot k) \leq (1 + C + \ldots + C^{k-1}) r_{f(x)^k}(n).$$

Polynomial	λ	d	Polynomial	λ	d
$1+x$	3	1	$1+x+x^6$	3.45686	20
$1+x+x^2$	3.23607	2	$1+x+x^2+x^6$	3.49009	20
$1+x+x^3$	3.31142	4	$1+x+x^3+x^6$	3.50478	10
$1+x+x^4$	3.33159	5	$1+x^2+x^3+x^6$	3.53521	20
$1+x+x^2+x^4$	3.3788	7	$1+x+x^2+x^3+x^6$	3.53141	19
$1+x+x^3+x^4$	3.47662	4	$1+x+x^4+x^6$	3.50468	17
$1+x+x^2+x^3+x^4$	3.45729	4	$1+x+x^2+x^4+x^6$	3.55002	19
$1+x+x^5$	3.35174	10	$1+x+x^3+x^4+x^6$	3.59415	16
$1+x^2+x^5$	3.46127	12	$1+x^2+x^3+x^4+x^6$	3.53665	15
$1+x+x^2+x^5$	3.49563	7	$1+x+x^2+x^3+x^4+x^6$	3.59043	11
$1+x+x^3+x^5$	3.45469	12	$1+x+x^5+x^6$	3.54536	14
$1+x^2+x^3+x^5$	3.46639	5	$1+x+x^2+x^5+x^6$	3.50809	18
$1+x+x^2+x^3+x^5$	3.5229	14	$1+x+x^2+x^3+x^5+x^6$	3.57066	17
$1+x+x^2+x^4+x^5$	3.47168	11	$1+x+x^2+x^4+x^5+x^6$	3.49995	6
$1+x+x^2+x^3+x^4+x^5$	3.52951	6	$1+x+x^2+x^3+x^4+x^5+x^6$	3.5598	6

Table 4: λ and the degree of its minimal polynomial for $p=2$ and $\deg(f(x)) \leq 6$

This implies that $\lambda(f) \leq \lambda(f^k)$. Similarly since $q_{f(x)}(j \cdot k - i) \geq C^{-i} q_{f(x)}(j \cdot k)$, we can show that $\lambda(f^k) \leq \lambda(f)$. Therefore $\lambda(f) = \lambda(f^k)$. □

3.3.1 Values of λ where $p = 2$

We calculate λ for polynomials with $p = 2$. We also find the minimal polynomial of λ. Provided are λ and the degree d of its minimal polynomial for non-similar polynomials with degree of up to 6, although we had calculated for $deg(f) \leq 9$.

We see that λ is between 3 and 4. We form several conjectures on the bounds of λ.

Conjecture 3.7. *When $p = 2$, $\lambda \geq 3$. Furthermore, $\lambda = 3$ only for polynomials similar to $1+x$. If $p = 2$ and $\lambda > 3$, then $\lambda \geq 1+\sqrt{5}$. Furthermore, $\lambda = 1+\sqrt{5}$ only if $f(x)$ is similar to $1+x+x^2$.*

Question 3.8. *Is it true that $\lambda(f) = \lambda(g)$ if and only if $f(x)$ and $g(x)$ are similar in terms of the transformations described in Theorem 3.6?*

Theorem 3.9. *For some polynomial $f(x)$ with degree at most 2^k and $p = 2$,*
$\lambda(f) \leq 4(1 - \frac{1}{2^{k+2}})^{\frac{1}{k+1}}$.

Proof. Define k such that the degree of $f(x)$ is at most 2^k, with $p = 2$. From Theorem 3.3, we can draw an oriented graph whose vertices are elements of X and whose edges correspond to the four maps. Therefore there are exactly four edges coming out of each vertex. Therefore if $Q(n)$ is the number of paths in the graph of length n, we have $\log \lambda = \limsup_{n\to\infty} \frac{\log Q(n)}{n}$. From the definition of Willson's method, Theorem 3.1, two of the four mappings correspond to $g(x) \to g(x^2)$ and $g(x) \to x \cdot g(x^2)$. Assume $\deg(f(x)) = 2^k$. Then a path starting from any $g(x)$ and moving first to $x \cdot g(x^2)$ then alternating in any way between the two mappings leads to 0 after $k+1$ steps. So the number of such paths of length $k+1$ is 2^k. So the number of paths of length $k+1$ from any point that avoids 0 is at most $4^{k+1} - 2^k$. Thus the number of such paths of length $n \cdot (k+1)$ is at most $(4^{k+1} - 2^k)^n$. This gives us the bound of $\lambda \leq 4(1 - \frac{1}{2^{k+2}})^{\frac{1}{k+1}}$. □

For $k = 0$, the only polynomial is $1 + x$, so the bound $\lambda \leq 4(1 - \frac{1}{4})^1 = 3$ is sharp. However, for $k = 1$ the bound tells us that $\lambda \leq \sqrt{14}$ which is not sharp. Furthermore, this bound approaches 4 as k approaches ∞.

Conjecture 3.10. *Let Λ_k be the maximal $\lambda(f)$ for deg $f \leq k$. Then $lim_{k\to\infty}\Lambda_k = 4$.*

Remark 3.11. *Similarly for $p > 2$, one may conjecture that $lim_{k\to\infty}\Lambda_k = p^2$.*

Through computer analysis of λ for $p = 2$ and $deg\Big(f(x)\Big) \leq 9$, Conjecture 3.12 arises.

Conjecture 3.12. *The degree of the minimal polynomial of λ is less than or equal to $2^{deg(f)-1}$ for $p = 2$.*

4 Conclusion and Directions of Future Research

Natural goals for further study of the phenomena examined in this paper include the following:

- Obtain recursion relations, generating functions, and limiting functions as in Section 2 for $a_{f(x),p}(n)$ in the case $deg\Big(f(x)\Big) > 1$;
- Prove Conjecture 2.13 on the functional equation for the generating function for $a_{f(x),p}(n)$;

- Prove the conjectures in section 3 on the behavior of the eigenvalues λ and obtain better upper bounds;
- Find, tighten, and explore the upper bound mentioned in Conjecture 3.12;
- Study the algebras generated by the four transformations composing the Willson matrices and find analogs for larger p.

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