A Study of Bar and Arc k-Visibility Graphs

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1 Research Experience

It is actually impossible to explain my experience in math research without beginning with my experience in math contests. As a relatively accomplished contestant over my high school years, including participating the United States of America Junior Mathematical Olympiad (USAJMO) and twice in United States of America Mathematical Olympiad (USAMO), I fell in love with the mathematics and the often slick and beautiful solutions in these contests. However math contest can be incredibly deceiving as in most serious mathematics the necessary background knowledge can be quite cumbersome for high school students. Research in analytic number theory, my mentor once joked requires a PhD to understand. But in modern mathematics there is at least one notable exception, combinatorics.

My path to combinatorics research was rather circuitous. I initially did solar panel research under the tutelage of my research teachers Mr. Kurtz, Dr. Solomon, and Mrs. Collette with a fellow future Intel Semifinalist, David Li. As a group we initially developed the idea and then implemented a system for collecting and then processing thousands of data points on power production and temperature. Although this research was published, I had already fallen head over heels in love for mathematics. In this case my research teacher Mr. Kurtz, after a year doing topology research at Stony Brook University, recommended I apply for MIT PRIMES program where I was able to do combinatorics research.

MIT PRIMES-USA Program, a section of 15 students nationwide, is a group of students who conduct mathematics research with MIT graduate students or faculty through the use of FaceTime or Skype. Through this program I was selected to work with Jonathan Weed, who at the time was a first year graduate student at MIT and the topic of Bar and Arc k-Visibility Graphs was suggested by Jesse Geneson. My mentor initially guided me through the background by providing me with the seminal papers in the research area and discussing these with me every week. Although the papers were quite difficult, the technical aspects of these papers were well within my reach just from the problems I had done in olympiads, and instead the difficultly lay in elaborate and clever nature of these arguments. This is largely a function of the fact that number theory and "geometry" have been studied since antiquity but modern combinatorics is largely due to the innovations of Paul Erdos since the 1930's and 1940's.

The research phase of the project proved difficult. Although I (and my mentor) made limited progress, the initial approach of using a technique of graph minors was remarkably difficult and was only used in one of the seven sections in my paper. The key innovation in the project came when I was sitting in my chemistry class. Bored out of my mind I began using regular polygons to generate a set of graphs, they had the odd property of having arbitrarily long induced cycles. This was unusual as both me and my mentor had conjectured that such graphs don't exist but the construction at first was simply an odd anomaly.

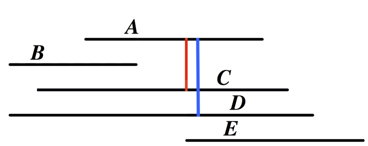
This still surprises me to this day, but a look back at this construction, with a slight modification allowed me to show that an edge bound for SemiArc k-Visibility Graphs is optimal, disproving a pair of open conjectures in the process. The surprises with the configuration didn't end here. Using this as inspiration it followed that all SemiArc Visibility Graphs are planar, and subsequently I gave a full classification of such graphs using this result.

Although the "utility" of the initial configuration ended at this point, this provided a sense of confidence that was extremely useful. Within a period of three or four weeks I proved nearly all of the remaining results in my paper, with the majority of results addressing unanswered questions or extending previous results. However none of this would have been possible without my mentor's encouragement in helping me select which ideas appeared useful and which questions to pursue.

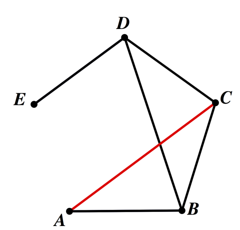
This did not simply conclude my research. The remainder of the time between this end in active research and my Intel submission involved extensive drafting of my research and then creating a presentation for my local affiliate fair for the Intel Science and Engineering Fair (ISEF). Through the extensive help of both my mentor and research teachers, Mr. Kurtz, Dr Solomon, Mr. Collette, and Dr. Kramer, I was able to both check over each of the proofs and then craft a paper that explained the necessary background and applications in an intelligible way to a non-mathematical audience. Without their help I simply would not have been selected as an Intel Semifinalist, Intel Science and Engineering Fair Finalist, and an Outstanding Presenter at the MAA Undergraduate Poster Session.

2 Research Section

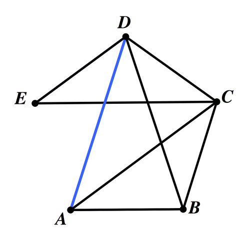
Given the scope of results in this research I will focus on two particular results that highlight the results which are of greatest interest. However it is first necessary to define Bar k-Visibility Graphs, Arc k-Visibility Graphs, SemiBar k-Visibility Graphs, and SemiArc k-Visibility Graphs.



(a) Visibility representation



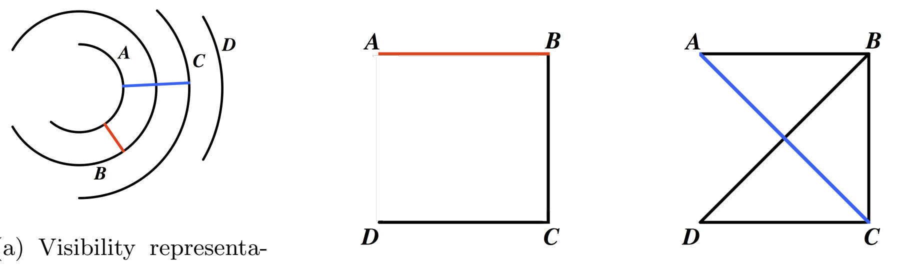
(b) Visibility graph



(c) 1-visibility graph

Figure 1: Bar (k-)visibility

Bar (k-)visibility graphs are defined by taking the regions to be nonintersecting closed horizontal line segments in the plane ("bars") connected by vertical lines of sight. Requiring lines of sight to be unobstructed yields bar visibility graphs; allowing them to intersect up to k additional bars yields bar k-visibility graphs. Figure 1 shows a collection of bars and the corresponding visibility and 1-visibility graphs.



(a) Visibility representation

(b) Visibility graph

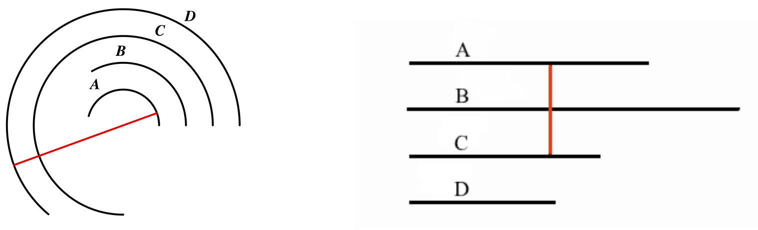
(c) 1-visibility graph

Figure 2: Arc (k-)visibility

An extension arc (k-)visibility graphs were introduced by Hutchinson [4] and Dean et al. [2]. These are defined by taking the regions to be nonintersecting concentric circular arcs and lines of sight to be radial line segments, which may pass through the center of the circle. The notion of visibility and k-visibility remain exactly the same when transferring from the context of bar (k-)visibility graphs to arc (k-)visibility graphs. Examples of arc (k-)visibility graphs appear in Figure 2.

Finally, I considered two important special cases of the classes defined above. Semi-bar visibility graphs, introduced by Felsner and Massow [3], are bar visibility graphs where the left endpoints of all the bars lie on the same vertical line. Likewise, in semi-arc visibility graphs, introduced by Babbitt et al. [1], arcs extend in a counterclockwise direction from the same radial ray. Figure 3 gives examples of semi-bar and semi-arc visibility representations.

The first theorem I will introduce provides the first step in giving a classification of all semi-arc visibility graphs.



(a) Semi-arc visibility representation (b) Semi-bar visibility represenation

Figure 3

Theorem 1. *All semi-arc visibility graphs are planar.*

The proof of this theorem will be explained informally. First note that it suffices to consider the positions where all arcs have different left radial endpoints, as by perturbing arcs with the same radial endpoints the number of edges does not decrease. Then using a series of reductions, which are rather technical, it suffices to consider the case where all arcs have increasing angular argument. To further simplify the arrangement, each endpoint of an arc was then approximated by a sufficiently large regular n-gon and using graph minors it is possible to add arcs corresponding to vertices which are not covered. Finally this set of "regular" configurations can be shown to be planar by explicitly drawing the desired graph and this completes the proof.

Although the proof as explained may appear to some extent, pulled out of thin air, this was largely motivated by noticing that the configuration with the most edges, and therefore empirically least likely to be planar, were the highly regular configurations. And this sketch is a natural implementation of the idea.

The second theorem concerns the expected number of edges (and the variance in the expected number of edges proved by my mentor) in a random semi-bar k-visibility graph. Random here is defined as setting the left endpoint to 0 and varying the right endpoint at random from 0 to 1 with any i.i.d. (or for nontechnical readers, uniformly at random.) Due to the relative simplicity of the argument it will provided in full.

Theorem 2. *Let G be a random semi-bar k-visibility graph as defined above and let E be its number of edges. Then $\mathbb{E}[E] = \binom{n}{2}$ for $n \le k+2$ and*

$$\mathbb{E}[E] = \frac{1}{2}(k+1)\left(4n - 3k - 6 - 2(k+2)\sum_{l=k+3}^{n}\frac{1}{l}\right) = (k+1)(2n - o(n))$$

for $n \ge k+3$. Moreover, for any $t \ge 0$,

$$\mathbb{P}(|E - \mathbb{E}[E]| > (k+1)t) \le 2\exp\left(-\frac{2t^2}{n}\right).$$

Proof. If $n \le k+2$, then G is the complete graph and the claims are trivial. So suppose that $n \ge k+3$.

Since creating a random semi-bar k-visibility graphs is equivalent to drawing a permutation uniformly at random, we can generate G by generating a permutation one element at a time. In each of n rounds, we add a bar, shorter than all those added thus far, to a semi-bar visibility representation in a random position. Bars added in this way do not affect the visibilities already present in the graph, so it suffices to consider those added by the addition of the new bar.

In general, the addition of a new bar adds $2k+2$ edges, except when the new bar has fewer than $k+1$ bars to its right or left. If m bars have already been added, then there are $m+1$ possible positions for the new bar, each equally likely. If $m \le k+1$, then all placements of the new bar add m edges. If $m \ge k+2$, then the addition of the new bar adds between $k+1$ and $\min\{m, 2k+2\}$ edges. In either case the difference between the largest and smallest possible number of additional edges is at most $k+1$. Applying the Azuma-Hoeffding inequality yields the concentration bound.

To find the expected number of edges, we apply linearity of expectation. Suppose m bars have been added so far. If $m \le k+1$, then as noted above the expected number of edges associated with the new bar is m. If $m > k+1$, the expected number of edges between the new bar and bars to its right is $\frac{1}{m+1}\left((m-k)(k+1) + \sum_{\ell=0}^{k}\ell\right)$, and by symmetry the total expected number of new edges is twice this number. Summing and simplifying yields the desired bound. □

References

[1] Matthew Babbitt, Jesse Geneson, and Tanya Khovanova, *On k-visibility graphs*, J. Graph Algorithms Appl. **19** (2015), no. 1, 345–360. MR 3376666

[2] Alice M. Dean, William Evans, Ellen Gethner, Joshua D. Laison, Mohammad Ali Safari, and William T. Trotter, *Bar k-visibility graphs*, J. Graph Algorithms Appl. **11** (2007), no. 1, 45–59. MR 2318424 (2008g:05139)

[3] Stefan Felsner and Mareike Massow, *Parameters of bar k-visibility graphs*, J. Graph Algorithms Appl. **12** (2008), no. 1, 5–27. MR 2392932 (2009e:68094)

[4] Joan P. Hutchinson, *Arc- and circle-visibility graphs*, Australas. J. Combin. **25** (2002), 241–262. MR 1884050 (2002m:05150)